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LIQUIDITY IN ASSET MARKETS WITH SEARCH FRICTIONS

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LIQUIDITY IN ASSET MARKETS WITH SEARCH FRICTIONS

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We develop a search-theoretic model of financial intermediation in an over-the-counter market and study how trading frictions affect the distribution of asset holdings and standard measures of liquidity. A distinctive feature of our theory is that it allows for unrestricted asset holdings, so market participants can accommodate trading frictions by adjusting their asset positions. We show that these individual responses of asset demands constitute a fundamental feature of illiquid markets: they are a key determinant of trade volume, bid–ask spreads, and trading delays—the dimensions of market liquidity that search-based theories seek to explain.

KEYWORDS: Bid–ask spreads, trading delays, liquidity, search, trade volume.

1. INTRODUCTION

RECENT LITERATURE pioneered by Duffie, Gârleanu, and Pedersen (2005) (DGP) uses search theory to model the trading frictions that are characteristic of over-the-counter (OTC) markets.[2] The search-based approach is appealing because it can parsimoniously rationalize standard measures of liquidity such as trade volume, bid–ask spreads, and trading delays, and can be used to study how market conditions influence these measures. A virtue of DGP's formulation is that it is analytically tractable, so all these mechanisms can be well understood.

The literature spurred by DGP keeps the framework tractable by imposing a stark restriction on asset holdings: agents can only hold either 0 units or 1 unit of the asset. In effect, investors' ability to respond to changes in market conditions is severely limited by this restriction. In this paper we develop a search-based model of liquidity in asset markets with no restrictions on investors' asset holdings. The model is close in structure and spirit to DGP, but captures the heterogeneous responses of individual investors to changes in market conditions.

As a result of the restrictions they imposed on asset holdings, existing search-based theories neglect a critical feature of illiquid markets, namely, that mar-

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[2]The search-theoretic literature on financial markets also includes Duffie, Gârleanu, and Pedersen (2007), Gârleanu (2008), Miao (2006), Rust and Hall (2003), Spulber (1996), and Weill (2007).

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ket participants can mitigate trading frictions by adjusting their asset positions to reduce their trading needs.[3] The key theoretical observation is that an investor's asset demand in an OTC market depends not only on his valuation for the asset at the time of the trade, but also on his expected valuation over the holding period until his next opportunity to trade. A reduction in trading frictions makes investors less likely to remain locked into an undesirable asset position and therefore induces them to put more weight on their current valuation. As a result, a reduction in trading frictions induces an investor to demand a larger asset position if his current valuation is relatively high and a smaller position if it is relatively low, which tends to increase the spread of the distribution of asset holdings. We find that this effect on the dispersion of the distribution of asset holdings is a key channel through which trading frictions determine trade volume, bid–ask spreads, and trading delays—the dimensions of market liquidity that search-based theories of financial intermediation are designed to explain.

Trade volume is a manifestation of the ability of the exchange mechanism to reallocate assets across investors. We find that by increasing the dispersion of asset positions, a reduction in trading delays (or in dealers' market power) tends to increase trade volume. Bid–ask spreads constitute the main out-of-pocket transaction cost in an illiquid market. Our model generates a distribution of spreads across trade sizes and predicts that spreads per unit of asset traded decrease with the ease with which investors can find alternative trading partners (a mechanism identified in DGP), but increase with the size of the trade. Since reduced trading delays tend to increase trade sizes, marketwide measures of transaction costs can vary in a nonmonotonic fashion with the extent of the trading frictions. Trading delays are a distinguishing feature of an OTC market. We find that the distribution of asset holdings is a key determinant of trading delays and that the interaction with the dealers' incentives to make markets generates a liquidity externality that can give rise to multiple steady states.

There is a connection between DGP and Kiyotaki and Wright (1989): both are search-based theories of exchange, and both restrict asset holdings to keep the distribution of assets manageable. Similarly, there is also a connection between our work and the monetary literature that attempts to generalize the inventory restrictions of Kiyotaki and Wright (e.g., Camera and Corbae (1999) and Molico (2006)). In standard monetary models, idiosyncratic trading shocks are the only source of heterogeneity, so in his numerical examples, Molico (2006) found that the distribution of money holdings becomes more concentrated as trading frictions are reduced. Our theory has the opposite prediction

[3]The importance of this mechanism in the context of another class of models—those with exogenous transaction costs—has been stressed by Constantinides (1986) for the case of proportional transaction costs, and by Lo, Mamaysky, and Wang (2004) for the case of fixed transaction costs.

due to the asset reallocation that dealers carry out among investors with heterogeneous valuations. Another difference between our work and the analogous monetary literature is that, aside from a few exceptions (e.g., Green and Zhou (2002)), the latter is eminently computational, and theoretical results are limited. Recent search models of money (e.g., Lagos and Wright (2005)) allow for unrestricted inventories but keep the analysis tractable by making assumptions that render the distribution of money holdings degenerate. In contrast, the heterogeneity in asset holdings that is propagated endogenously by random matching is an important feature of our model, and we are able to provide an analytical characterization of the equilibrium—including transitional dynamics and the endogenous distribution of asset holdings.

2. ENVIRONMENT

Time is continuous, starts at $t = 0$, and goes on forever. There are two types of infinitely lived agents: a unit measure of investors and a unit measure of dealers. There is one asset, one perishable consumption good called *fruit*, and another consumption good defined as *numéraire*. The asset is durable, perfectly divisible, and in fixed supply, $A \in \mathbb{R}_+$. Each unit of the asset produces a unit flow of fruit. There is no market for fruit, so holding the asset is necessary to consume this good. The numéraire good is produced and consumed by all agents. The instantaneous utility function of an investor is $u_i(a) + c$, where $a \in \mathbb{R}_+$ represents the fruit consumption (which coincides with the investor's asset holdings), $c \in \mathbb{R}$ is the net consumption of the numéraire good ($c < 0$ if the investor produces more of these goods than he consumes), and $i \in \mathbb{X} = \{1, \dots, I\}$ indexes a preference type. The utility function $u_i(a)$ is twice continuously differentiable, strictly increasing, and strictly concave.[4] Each investor receives a preference shock with Poisson arrival rate δ. This process is independent across investors. Conditional on the preference shock, the probability the investor draws preference type i is $\pi_i > 0$, with $\sum_{i=1}^{I} \pi_i = 1$. These preference shocks capture the notion that investors will value the asset differently over time, thereby generating the need to rebalance their asset posi-

[4]Just as in DGP, our specification associates a certain utility to the investor as a function of his asset holdings. The utility from holding an asset position could be simply the value from enjoying the asset itself, as would be the case for real assets such as cars or houses. An alternative interpretation that leads to the same formulation would be to assume that there is a single consumption good, that investors are risk-neutral and able to borrow and lend freely at rate r, and regard the asset as physical capital used to produce the consumption good with the production technology u_i. As yet another possibility, one could adopt the preferred interpretation of DGP, namely that u_i is in fact a reduced-form utility function that stands in for the various reasons why investors may want to hold different quantities of the asset, such as differences in liquidity needs, financing or financial-distress costs, correlation of asset returns with endowments (hedging needs), or relative tax disadvantages. By now, several papers that build on the work of DGP have formalized the "hedging needs" interpretation. Examples include Duffie, Gârleanu, and Pedersen (2007), Gârleanu (2008), and Vayanos and Weill (2008).

tions.[5] Dealers do not hold positions and their instantaneous utility is c, their consumption of the numéraire good.[6] All agents discount at rate $r > 0$.

Dealers can trade the asset continuously in a competitive interdealer market. Investors periodically contact dealers who can trade in this market on their behalf. Meetings with dealers occur at random according to a Poisson process with arrival rate α.[7] Once a dealer and an investor have contacted each other, they negotiate the quantity of assets that the dealer will acquire for the investor and the intermediation fee that the dealer charges for his services. After the transaction has been completed, the dealer and the investor part ways.

Asset holdings and preference types lie in the sets $\mathbb{R}_+$ and $\mathbb{X}$, respectively, and vary across investors and over time. We describe this heterogeneity with a probability space $(\mathbb{S}, \Sigma, H_t)$, where $\mathbb{S} = \mathbb{R}_+ \times \mathbb{X}$, Σ is the σ-field generated by the sets $(\mathcal{A}, \mathcal{I})$, where $\mathcal{A} \subseteq \mathbb{R}_+$ and $\mathcal{I} \subseteq \mathbb{X}$, and H_t is a probability measure on Σ that represents the distribution of investors across asset holdings and preference types at time t.

3. EQUILIBRIUM

Let $V_i(a, t)$ denote the maximum expected discounted utility attainable by an investor who has preference type i and is holding a assets at time t. The value function $V_i(a, t)$ satisfies

$$V_i(a,t) = \mathbb{E}_i\Bigg[\int_t^{T_\alpha} e^{-r(s-t)} u_{k(s)}(a)\,ds + e^{-r(T_\alpha - t)}\big\{V_{k(T_\alpha)}\big[a_{k(T_\alpha)}(T_\alpha), T_\alpha\big] - p(T_\alpha)\big[a_{k(T_\alpha)}(T_\alpha) - a\big] - \phi_{k(T_\alpha)}(a, T_\alpha)\big\}\Bigg], \tag{1}$$

where T_α denotes the next time the investor contacts a dealer and $k(s) \in \mathbb{X}$ denotes the investor's preference type at time s. The expectations operator, $\mathbb{E}_i$, is over the random variables T_α and $k(s)$, and is indexed by i to indicate that it is conditional on $k(t) = i$.[8] The first term on the right side of (1) contains the expected discounted utility flows over the time interval $[t, T_\alpha]$, whose

[5]In online Appendix B (Lagos and Rocheteau (2009)), we allow preference shocks to follow a general continuous-time Markov chain and find that most of the substantive results generalize under appropriate regularity conditions.

[6]The restriction that dealers cannot hold assets is immaterial when analyzing steady-state equilibria. Lagos, Rocheteau, and Weill (2007) studied dynamic equilibria where dealers may choose to hold asset positions.

[7]Although our description of the trading process is stylized, it captures the salient features of the actual trading arrangements in OTC markets. We refer the interested reader to Schultz (2001) as well as the discussion in Section 2.1 in Lagos and Rocheteau (2006).

[8]For now we proceed under the assumption that the right side of (1) is well defined. Later in this section we verify that this is the case by calculating $V_i(a, t)$ explicitly. More generally, in

length is exponentially distributed with mean $1/\alpha$. The flow utility is indexed by the preference type, $k(s)$, which follows a compound Poisson process with $\Pr[k(s)=j|k(t)=i]=[1-e^{-\delta(s-t)}]\pi_j+e^{-\delta(s-t)}\mathbb{I}_{\{j=i\}}$ for $s\geq t$. The second term on the right side of (1) is the expected discounted utility from the time when the investor next contacts a dealer, T_α, onward. At this time T_α, the dealer purchases $a_{k(T_\alpha)}(T_\alpha)-a$ in the market (or sells if this quantity is negative) at price $p(T_\alpha)$ on behalf of the investor; the investor readjusts his asset holdings from a to $a_{k(T_\alpha)}(T_\alpha)$ and pays the dealer an intermediation fee $\phi_{k(T_\alpha)}(a,T_\alpha)$. Throughout, we will focus on price functions $p(t)$ that are nonnegative and Lebesgue measurable. Both the fee and the asset price are expressed in terms of the numéraire good.[9]

Let $W(t)$ denote the maximum expected discounted utility attainable by a dealer. It satisfies

$$W(t)=\mathbb{E}\left\{e^{-r(T_\alpha-t)}\left[\int_{\mathbb{S}}\phi_i(a,T_\alpha)\,dH_{T_\alpha}+W(T_\alpha)\right]\right\},$$

where the expectations operator, $\mathbb{E}$, is over the next time the dealer meets an investor, T_α. Random matching implies that the investor whom the dealer meets is a random draw from H_{T_α}, the distribution of investors across preference types and asset holdings at time T_α.

We turn to the determination of the terms of trade in a bilateral meeting at time t between a dealer and an investor of type i who is holding a. Let a' denote the investor's posttrade asset holdings and let ϕ denote the intermediation fee. We take (a',ϕ) to be the outcome corresponding to the Nash solution to a bargaining problem where the dealer has bargaining power $\eta\in[0,1]$. The utility of the investor is $V_i(a',t)-p(t)(a'-a)-\phi$ if an agreement (a',ϕ) is reached and is $V_i(a,t)$ in case of disagreement. Therefore, the investor's gain from trade is $V_i(a',t)-V_i(a,t)-p(t)(a'-a)-\phi$. Analogously, the utility of the dealer is $W(t)+\phi$ if an agreement (a',ϕ) is reached and is $W(t)$ in case of disagreement, so the dealer's gain from trade is the fee, ϕ. The bargaining outcome is

$$\text{(2)}\qquad [a_i(t),\phi_i(a,t)] =\arg\max_{(a',\phi)}[V_i(a',t)-V_i(a,t)-p(t)(a'-a)-\phi]^{1-\eta}\phi^\eta,$$

online Appendix D (Lagos and Rocheteau (2009)) we formulate the investor's infinite-horizon problem from the time-0 perspective, and formalize the relationship between the maximum value of that problem and the function $\{V_i\}_{i\in\mathbb{X}}$ that satisfies (1).

[9]Since the intermediation fee is determined in a bilateral meeting, it may depend on the investor's preference type and asset holdings. Our notation for the investor's new asset position, $a_{k(T_\alpha)}(T_\alpha)$, makes explicit that it may depend on time and on the investor's preference type at the time of the trade. Below (condition (3)), we will find that the investor's new asset position is independent of the asset position he was holding at the time of the trade. To simplify the notation, we anticipate this result and do not include a as an argument of his new asset position.

where the maximization is subject to $a' \geq 0$.[10] The solution (2) can be written as

$$a_i(t) = \arg\max_{a' \geq 0}[V_i(a', t) - p(t)a'], \tag{3}$$

$$\phi_i(a, t) = \eta\{V_i[a_i(t), t] - V_i(a, t) - p(t)[a_i(t) - a]\}. \tag{4}$$

We now turn to the investor's problem. Substitute (3) and (4) into (1) to obtain

$$\begin{aligned} V_i(a, t) = \mathbb{E}_i\bigg[\int_t^{T_\alpha} & e^{-r(s-t)} u_{k(s)}(a)\,ds \\ & + e^{-r(T_\alpha - t)}\Big\{(1-\eta)\max_{a' \geq 0}\big[V_{k(T_\alpha)}(a', T_\alpha) - p(T_\alpha)(a' - a)\big] \\ & \qquad + \eta V_{k(T_\alpha)}(a, T_\alpha)\Big\}\bigg]. \end{aligned} \tag{5}$$

It is apparent from (5) that the investor's payoff is the same he would get in an alternative environment where he meets dealers according to a Poisson process with arrival rate α, but instead of bargaining, he readjusts his asset position and extracts the whole surplus with probability $1 - \eta$, whereas with probability η he cannot readjust his asset position and enjoys no gain from trade. Therefore, from the standpoint of the investor, keeping the paths of the aggregate variables unchanged, the environment we are analyzing is payoff-equivalent to an alternative one in which he meets dealers according to a Poisson process with arrival rate $\kappa = \alpha(1 - \eta)$ and has all the bargaining power in bilateral negotiations. Based on this observation, the following lemma offers an equivalent formulation of the investor's choice of asset holdings that appears on the right side of (5).

[10]The maximum in (2) is achieved provided that $\max_{a'}[V_i(a', t) - p(t)a']$ is achieved, which will be the case in equilibrium (see Lemma 1 and the proof of Proposition 1). Also, note that it would be equivalent to set $\phi = (\hat{p} - p(t))(a' - a)$ in (2) and reformulate the bargaining problem as a choice of $(a' - a, \hat{p})$. If $a' > a$, the investor is a buyer and $\hat{p} > p(t)$ can be interpreted as the *ask price* he is charged by the dealer. Conversely, if $a' < a$, the investor is a seller and $\hat{p} < p(t)$ is the *bid price* he is paid by the dealer. The outcome from the axiomatic Nash solution can also be obtained from a strategic bargaining game in which, upon contact, a randomly selected proposer makes a take-it-or-leave-it offer. Nature selects the dealer to make an offer with probability η, which the investor must either accept or reject on the spot. With complement probability $1 - \eta$, the investor makes the offer and the dealer either accepts or rejects on the spot. It is easy to check that the expected equilibrium outcome of this game coincides with (2), subject to the obvious reinterpretation of $\phi_i(a, t)$ as an *expected* intermediation fee, which is inconsequential. See online Appendix C (Lagos and Rocheteau (2009)) for details.

LEMMA 1: *An investor with preference type i and asset holdings a who readjusts his asset position at time t chooses*

$$a_i(t) = \arg\max_{a' \geq 0}[\bar{u}_i(a') - q(t)a'], \tag{6}$$

where

$$\bar{u}_i(a) = \frac{(r+\kappa)u_i(a) + \delta \sum_j \pi_j u_j(a)}{r+\kappa+\delta}, \tag{7}$$

$$q(t) = (r+\kappa)\left[p(t) - \kappa \int_0^\infty e^{-(r+\kappa)s} p(t+s)\,ds\right]. \tag{8}$$

If $q(t) > \bar{u}_i'(\infty)$, then $a_i(t)$ exists and is unique.

In Lemma 1, $\bar{u}_i(a)/(r+\kappa)$ is the expected discounted utility and $q(t)/(r+\kappa) = p(t) - \mathbb{E}[e^{-r(T_\kappa - t)} p(T_\kappa)]$ is the present value of the expected capital loss to the investor from holding a from t until the next (*effective*) time T_κ when he readjusts his holdings, where $T_\kappa - t$ is exponentially distributed with mean $1/\kappa$. The assumptions on $q(t)$ are without loss of generality since they will be implied by the market-clearing condition. Given that u_i is strictly concave for each i, the asset position $a_i(t)$ solves the maximization problem on the right side of (6) at time t if and only if it satisfies

$$\bar{u}_i'[a_i(t)] \leq q(t) \quad \text{“}=\text{” if } a_i(t) > 0. \tag{9}$$

In online Appendix D (Proposition 9) we show that a feasible asset plan $\{(a_i(t), t \in [0,\infty))\}_{i=1}^I$ maximizes the investor's infinite-horizon problem from the time-0 perspective if and only if it satisfies (9) and

$$\lim_{n\to\infty} \mathbb{E}_i\big[e^{-rT_n} p(T_n) a_{k(T_n)}(T_n)\big] = 0, \tag{10}$$

where T_n, for $n = 1, 2, \ldots,$ denotes the time at which the investor gains his nth effective access to the market. In online Appendix D we also establish a version of the principle of optimality for our economy (Lemma 8), and show that if there exist real numbers $\overline{B}$ and $\underline{B}$ such that $\max_j \bar{u}_j'(\infty) < \underline{B} \leq q(t) \leq \overline{B}$ for all t, then $V_i(a,t) = \bar{u}_i(a)/(r+\kappa) + [p(t) - q(t)/(r+\kappa)]a + K_i(t)$, where $K_i(t) \in \mathbb{R}$.

From (4), $\phi_i(a,t) = \eta\{V_i[a_i(t), t] - V_i(a,t) - p(t)[a_i(t) - a]\}$, with $a_i(t)$ characterized by (9). If we substitute the value function, we arrive at

$$\phi_i(a,t) = \frac{\eta\{\bar{u}_i[a_i(t)] - \bar{u}_i(a) - q(t)[a_i(t) - a]\}}{r+\kappa}. \tag{11}$$

Since each investor contacts a dealer with equal probability, the quantity of assets supplied in the interdealer market over a small interval of time dt is $\alpha\, dt\, A$.[11] Similarly, the measure of type-i investors who contact dealers is $\alpha\, dt\, n_i(t)$, where

$$n_i(t) = e^{-\delta t} n_i(0) + (1 - e^{-\delta t})\pi_i \tag{12}$$

is the measure of investors with preference type i at time t, so the demand for assets in the interdealer market is $\alpha\, dt \sum_{i=1}^{I} n_i(t) a_i(t)$. The clearing condition for the asset market is

$$\sum_{i=1}^{I} n_i(t) a_i(t) = A. \tag{13}$$

This condition implies that the $q(t)$ that clears the market is continuous and bounded.[12] Given such a $q(t)$ and (10), the following lemma shows how to recover $p(t)$.

LEMMA 2: *For any continuous and bounded $q(t)$, the price of the asset is*

$$p(t) = \frac{1}{r + \kappa}\left[q(t) + \kappa \int_t^{\infty} e^{-r(s-t)} q(s)\, ds\right]. \tag{14}$$

At any point in time, investors differ in asset holdings and preference types. Consider a set of asset holdings $\mathcal{A}$ and a set of preference types $\mathcal{I}$. Then for all $(\mathcal{A}, \mathcal{I}) \in \Sigma$, $H_t(\mathcal{A}, \mathcal{I})$ gives the measure of investors whose asset holdings and preference types lie in $\mathcal{A}$ and $\mathcal{I}$, respectively. We characterize this probability measure in the following lemma, where $\mathbb{I}_{\{a \in \mathcal{A}\}}$ denotes an indicator function that equals 1 if $a \in \mathcal{A}$.

LEMMA 3: *The measure of investors across individual states at time t satisfies*

$$H_t(\mathcal{A}, \mathcal{I}) = \sum_{i \in \mathcal{I}} \sum_{j=1}^{I} \left[n_{ji}^0(\mathcal{A}, t) + \int_0^t \mathbb{I}_{\{a_j(t-\tau) \in \mathcal{A}\}} n_{ji}(\tau, t)\, d\tau \right] \tag{15}$$

for all $(\mathcal{A}, \mathcal{I}) \in \Sigma$, where

$$n_{ji}^0(\mathcal{A}, t) = e^{-\alpha t}\left[(1 - e^{-\delta t})\pi_i + e^{-\delta t}\mathbb{I}_{\{i=j\}}\right] H_0(\mathcal{A}, \{j\}), \tag{16}$$

$$n_{ji}(\tau, t) = \alpha e^{-\alpha \tau}\left[(1 - e^{-\delta \tau})\pi_i + e^{-\delta \tau}\mathbb{I}_{\{i=j\}}\right] n_j(t - \tau). \tag{17}$$

[11] See Duffie and Sun (2007) for a derivation of the law of large numbers in random-matching environments.

[12] The asset demand, a_i, is a continuous function of q, and $n_i(t)$ is continuous in t, so $q(t)$ is continuous. Also, (13) implies $\max_i \bar{u}_i'(\infty) < q(t) \le \max_i \bar{u}_i'(A)$.

At time 0, the market starts with investors distributed across preference types and asset holdings according to the initial probability measure H_0. Subsequently, there are two types of investors: those who have not contacted a dealer since time 0 and those who have. The time-t measure of those who started at time 0 with preference type j and assets in $\mathcal{A}$, whose preference type is i at the current time t, and who have never traded (so their asset holdings are still in $\mathcal{A}$) is $n_{ji}^0(\mathcal{A}, t)$ as given in (16). Analogously, $n_{ji}(\tau, t)$ in (17) gives the time-t density of investors whose last trade was at time $t-\tau$ when their preference type was j and who have preference type i at time t.

DEFINITION 1: An equilibrium is a time path $\langle\{a_i(t)\}, q(t), p(t), \{\phi_i(a,t)\}, H_t\rangle$ that satisfies (6), (11), (13), (14), and (15), given an initial condition H_0.

PROPOSITION 1: *There exists a unique equilibrium. For any H_0, the equilibrium allocations and prices, $\langle\{a_i(t)\}, q(t), p(t), \{\phi_i(a,t)\}, H_t\rangle$, converge to the unique steady-state allocations and prices $\langle\{a_i\}, q, p, \{\phi_i(a)\}, H\rangle$ that satisfy $p = q/r$,*

$$\bar{u}_i'(a_i) \le q \quad \text{“}=\text{” if } a_i > 0, \tag{18}$$

$$\sum_{i=1}^{I} \pi_i a_i = A, \tag{19}$$

$$\phi_i(a) = \frac{\eta[\bar{u}_i(a_i) - \bar{u}_i(a) - q(a_i - a)]}{r+\kappa}, \tag{20}$$

$$H(\{a_i\}, \{j\}) = \frac{\delta\pi_i\pi_j + \alpha\pi_i\mathbb{I}_{\{i=j\}}}{\alpha+\delta}, \tag{21}$$

and $H(\mathcal{A}, \mathcal{I}) = 0$ for all $(\mathcal{A}, \mathcal{I}) \in \Sigma$ such that $\bigcup_{j=1}^{I}\{a_j\} \cap \mathcal{A} = \emptyset$.

It is possible to show that the equilibrium is efficient if and only if $\eta = 0$ (see online Appendix B). To illustrate how a reduction in trading delays affects the equilibrium, consider the limiting case $\kappa \to \infty$. From (7), $\bar{u}_i(a) \to u_i(a)$, and from (8) and (9), $u_i'[a_i(t)] \le q(t) = rp(t) - \dot{p}(t)$ for all i. From (13), $q(t) \to q^*(t)$, which solves $\sum_{i\in\mathcal{I}_t^+} n_i(t)u_i'^{-1}[q^*(t)] = A$, where $\mathcal{I}_t^+ = \{i \in \mathbb{X} : a_i(t) > 0\}$. From (11), $\phi_i(a,t) \to 0$ for all a, i, and t. With regard to the distribution of investors, $\alpha \to \infty$ implies that every investor holds his desired asset position at all times.[13] Thus, as frictions vanish, investors choose $a_i(t)$ continuously by equating their current marginal utility from holding the asset to its effective

[13]To see this, first note that (16) implies the measure of agents who have not contacted a dealer since time 0 vanishes; that is, $n_{ji}^0(\mathcal{A}, t) \to 0$ for all i and j, all t, and all $\mathcal{A} \subseteq \mathbb{R}_+$ as $\alpha \to \infty$. The time-t density of agents who have not contacted a dealer since time $t - \tau > 0$ is $n(\tau, t) = \sum_{i,j=1}^{I} n_{ji}(\tau, t)$. From (17), $\alpha \to \infty$ implies $n(\tau, t) \to 0$ for all $\tau > 0$, that is, investors can find a dealer instantly when α is arbitrarily large, so the measure of investors who have

cost $q^*(t)$, and the equilibrium fees, asset price, and distribution of asset holdings are the ones that would prevail in a Walrasian economy. In what follows, when we analyze the steady state we will denote an individual investor's state $(a_i, j) \in \{a_i\}_{i=1}^I \times \mathbb{X}$ by $(i, j) \in \mathbb{X}^2$ and $H(\{a_i\}, \{j\})$ by n_{ij}. Also, at times we use ϕ_{ji} to denote $\phi_i(a_j)$ for $(i, j) \in \mathbb{X}^2$.

4. SEARCH FRICTIONS AND THE DISTRIBUTION OF ASSET HOLDINGS

In this section we focus on the steady state to study the effects of trading frictions on the distribution of asset holdings. Hereafter we assume $u_i'(\infty) = 0$ and $u_i'(0) = \infty$ for each i.[14] Condition (18) becomes

$$\bar{u}_i'(a_i) = rp. \tag{22}$$

Let $a_i = g_i(\kappa; p)$ denote the choice of asset holdings characterized by (22). Then

$$\frac{\partial g_i(\kappa; p)}{\partial \kappa} = \frac{\delta \left[u_i'(a_i) - \sum_{j=1}^{I} \pi_j u_j'(a_i) \right]}{-\bar{u}_i''(a_i)(r + \kappa + \delta)^2} \tag{23}$$

has the sign of $u_i'(a_i) - \sum_{j=1}^I \pi_j u_j'(a_i)$, that is, an investor whose current marginal valuation exceeds his expected marginal valuation over the expected holding period increases his demand when κ increases. If $u_i'(a_i) > \sum_{j=1}^I \pi_j u_j'(a_i)$, the investor anticipates that his valuation is likely to revert toward $\sum_{j=1}^I \pi_j u_j'(a_i)$ in the future and that when this happens, he may be unable to rebalance his asset position for some time. Thus, from (22), his choice of a_i is lower than $u_i'^{-1}(rp)$, what he would choose in a world with no trading delays. If α increases, the investor is more likely to find a dealer faster; if η decreases, it will be cheaper for the investor to readjust his asset holdings once he finds a dealer. In both cases, the investor assigns more weight to current marginal utility from holding the asset relative to the expected value, so his demand increases. Conversely, an investor with a current marginal valuation that is below his expected marginal valuation over the holding period reduces his demand when κ increases.[15] All this seems to suggest that the distribution of

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not met a dealer between $t - \tau$ and t is zero for all $\tau > 0$. As for those investors who have met a dealer this "instant," from (17), $n_{ji}(0, t) = 0$ for $i \neq j$ and $n_{ii}(0, t) = n_i(t)$. Therefore, $H_t(\mathcal{A}, \mathcal{I}) \to \sum_{i \in \mathcal{I}} \mathbb{I}_{\{a_i(t) \in \mathcal{A}\}} n_i(t)$ as $\alpha \to \infty$, that is, every investor of type i holds $a_i(t)$ at every t.

[14]These conditions imply that the investor's problem has a solution for all $q > 0$ and that the nonnegativity constraints in (6) are slack at every date for every investor in the unique equilibrium. This will simplify the notation, but is otherwise inessential for our results.

[15]In online Appendix B we show that this insight does not rely on preference shocks being independently and identically distributed (i.i.d.). There we derive an expression analogous to (23)

asset holdings will spread out if frictions are reduced. However, this intuition is only partial because (23) keeps the equilibrium asset price constant. Next, we study the effect of trading frictions on asset prices—a necessary step to establish the general equilibrium effect of trading frictions on the distribution of asset holdings.

Let $u_i(a) = \varepsilon_i u(a)$. Then (22) becomes $\bar{\varepsilon}_i u'(a_i) = rp$, where $\bar{\varepsilon}_i = ((r+\kappa)\varepsilon_i + \delta\bar{\varepsilon})/(r+\kappa+\delta)$ and $\bar{\varepsilon} = \sum_{j=1}^{I} \pi_j \varepsilon_j$. For a given p, as κ increases, the demands of investors with relatively low valuations ($\varepsilon_i < \bar{\varepsilon}$) fall, while those of investors with high valuations ($\varepsilon_i > \bar{\varepsilon}$) rise. Whether an increase in κ causes the asset price to rise depends on the curvature of the individual demand for the asset as a function of $\bar{\varepsilon}_i$, that is, on the slope of $\partial a_i/\partial\bar{\varepsilon}_i = -[u'(a_i)]^2/[u''(a_i)rp]$. In Appendix A (Proposition 5) we show that $dp/d\kappa \geq 0$ if $[u'(a)]^2/[u''(a)rp]$ is decreasing in a.[16] The following proposition characterizes the general equilibrium effect of trading frictions on the dispersion of the distribution of asset holdings.

PROPOSITION 2: (i) *For all* $i \in \{1, \dots, I\}$, $a_i \to A$ *as* $r + \kappa \to 0$.
(ii) *Let* $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ *with* $\sigma > 0$. *An increase in* κ *causes the equilibrium distribution of asset holdings to become more dispersed.*

According to part (i) of Proposition 2, the dispersion of the distribution of asset holdings approaches zero as trading frictions become very severe, provided that investors are sufficiently patient. This result holds for general preferences and will be useful in our analysis of trade volume, transaction costs, and trading delays.[17]

when preference shocks follow a general Markov process, and we provide several sufficient conditions that allow us to sign $\partial g_i(\kappa, p)/\partial\kappa$. We show, for instance, that for κ sufficiently large, $\partial g_i(\kappa, p)/\partial\kappa > 0$ if and only if $u_i'(a_i) < \sum_{j=1}^{I} \pi_{ij} u_j'(a_i)$, where π_{ij} is the probability that an investor with preference type i draws type j conditional on his receiving a preference shock. This condition is equivalent to the condition in part (i) of Proposition 2 in Gârleanu (2008). See Proposition 6 in online Appendix B (Lagos and Rocheteau (2009)) for details.

[16]For example, if $u(a) = a^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$, then $dp/d\kappa < 0$ (> 0) if $\sigma > 1$ (< 1). If $u(a) = \log a$, then a_i is linear in $\bar{\varepsilon}_i$ and $dp/d\kappa = 0$. This particular result is reminiscent of the findings in Constantinides (1986), Gârleanu (2008), and Heaton and Lucas (1995) that the equilibrium asset price is not (much) affected by transaction costs. In online Appendix B, we show that this finding generalizes to the more general case of Markovian preference shocks.

[17]In online Appendix B (part (iii) of Proposition 8 in Lagos and Rocheteau (2009)), we show that part (i) of Proposition 2 also holds for more general preference shock processes. The proof of part (ii) of Proposition 2 relies on the assumption of i.i.d. preference shocks and its immediate mean-reverting property. The i.i.d. specification, however, is without loss of generality for the case $I = 2$. (This is the case analyzed by DGP and much of the subsequent literature.) For $I > 2$, an increase in trading frictions need not compress the cross-sectional distribution of asset holdings. As pointed out by Gârleanu (2008), it is possible that for certain ranges of κ, an investor with a high current valuation (relative to the cross section of current valuations) may increase his asset holdings in response to an increase in trading frictions. The general insight, however, is that

5. MARKET LIQUIDITY

In the previous section we showed that traders who operate in markets with OTC-style frictions will seek to mitigate these trading frictions by adjusting their asset positions so as to reduce their trading needs. In this section we show how this kind of "liquidity hedging" that we have identified—and that only becomes possible with unrestricted asset holdings—shapes the effects of trading frictions on the three key dimensions of market liquidity: trade volume, transaction costs, and trading delays.

Trade Volume

Let $\mathcal{V}$ denote trade volume, defined as

$$\mathcal{V} = \frac{\alpha}{2} \sum_{i,j=1}^{I} n_{ij} |a_j - a_i|. \tag{24}$$

An increase in α has three distinct effects on $\mathcal{V}$. First, the measure of investors in any individual state $(i, j) \in \mathbb{X}^2$ who gain access to the market and are *able to trade* increases, which tends to increase $\mathcal{V}$. Second, the proportion $1 - \sum_{i=1}^{I} n_{ii}$ of agents who are mismatched to their asset position—the fraction of agents who *wish to trade*—decreases, which tends to decrease $\mathcal{V}$. Finally, the distribution of asset holdings spreads out, which tends to increase the quantity of assets traded in many individual trades. With (21) and (24), it is possible to show that the first two effects combined lead to an increase in $\mathcal{V}$. Although it is difficult to sign the third effect in general due to the general equilibrium effects of the price on the distribution of asset holdings, the following proposition establishes analytical results for three cases.

PROPOSITION 3: (i) *Trade volume approaches zero as* $r + \kappa \to 0$.

(ii) *Let* $u_i(a) = \varepsilon_i \ln a$. *Trade volume increases with* κ. *Moreover, for any pair* (κ, κ') *such that* $\kappa' > \kappa$, *the distribution of trade sizes associated with* κ' *first-order stochastically dominates the one associated with* κ.

(iii) *Let* $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ *with* $\sigma > 0$ *and assume that* $I = 2$. *Trade volume increases with* κ.

Transaction Costs

Intermediation fees and the implied bid–ask spreads constitute the out-of-pocket transaction costs borne by investors and are commonly used measures

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investors always react to more severe trading frictions by choosing asset positions that reduce the expected sizes of their future asset reallocations.

of market liquidity.[18] At the same time, these spreads determine the revenue of dealers, and hence are a key determinant of their incentives to make markets and provide liquidity. Intermediation fees depend on the rate at which investors can contact alternative dealers, on their bargaining power in bilateral negotiations, and on the size of the trade. We showed in Propositions 2 and 3 that trade sizes tend to increase as trading frictions are reduced. The following result shows that, keeping the characteristics of an investor and a dealer constant, transaction costs—both total and per unit of asset traded—increase with the size of the trade.[19]

LEMMA 4: *Consider an investor who holds asset position* $a \geq 0$ *and wishes to trade* $|a_i - a| > 0$. *Both* $\partial\phi_i(a)/\partial a$ *and* $\partial/\partial a[\phi_i(a)/|a_i - a|]$ *have the same sign as* $a - a_i$.

In the general equilibrium, κ affects the distribution of asset holdings, and this can give rise to nonmonotonicities in trading costs in response to changes in the degree of trading frictions. We prove this result for the case of patient traders, both for intermediation fees for individual trades and for a marketwide measure of transaction costs, $\Phi = \sum_{i,j=1}^{I} n_{ji}\phi_{ji}$. The average fee, Φ, represents the expected revenue of an individual dealer conditional on meeting an investor.

PROPOSITION 4: (i) *For each* $(i, j) \in \mathbb{X}^2$, *there exists* $\bar{r} > 0$ *such that for all* $r < \bar{r}$ *and* $\eta \in (0, 1)$, ϕ_{ji} *is nonmonotonic in* κ *and is largest for some* $\kappa \in (0, \infty)$.

(ii) *There exists* $\hat{r} > 0$ *such that for all* $r < \hat{r}$ *and* $\eta \in (0, 1)$, Φ *is nonmonotonic in* κ *and is largest for some* $\kappa \in (0, \infty)$.

In very illiquid markets (as $r + \kappa \to 0$), investors hedge against future preference shocks by choosing asset holdings that reflect their average utility from holding the asset rather than their current utility at the time they trade. Thus, trade sizes and fees are small. In very liquid markets (as $\kappa \to \infty$) investors trade large quantities, but the fees they pay are also small because of favorable search options. For intermediate values of κ, trade sizes are considerable and dealers have a degree of market power that results in larger intermediation fees. Part (ii) of Proposition 4 implies that dealers are better off when

[18]See footnote 10 for the theoretical link between intermediation fees and bid–ask spreads.

[19]The theory generates a distribution of transaction costs, not only across trade-size categories, but also among trades of equal size, which is in accordance with the evidence from the OTC market for municipal bonds (Green, Hollifield, and Schurhoff (2007)). The increasing relationship between trade size and transaction cost for given α is consistent with the empirical evidence on foreign exchange markets (Burnside, Eichenbaum, Kleshchelski, and Rebelo (2006, Table 12)). In contrast, empirical studies on municipal and corporate bond markets document that larger trades tend to be executed at a discount (Harris and Piwowar (2006)). Our model can rationalize this observation if we allow for heterogeneous investors, some of which can contact dealers faster than others. See Lagos and Rocheteau (2006).

they trade in markets that are neither too liquid nor too illiquid. If κ is very large, dealers would find it profitable to shift the trading activity to markets with larger η or smaller α. Conversely if κ is very small, perhaps surprisingly, dealers would benefit from reductions in η or increases in α.

Trading Delays

Here we allow for free entry of dealers so as to endogenize the length of the trading delays and formalize the notion that a dealer's profit depends on the competition for order flow that he faces from other dealers. Let α now be a continuously differentiable function of the measure of dealers in the market, υ, with $\partial\alpha(\upsilon)/\partial\upsilon > 0$, $\partial[\alpha(\upsilon)/\upsilon]/\partial\upsilon < 0$, $\alpha(0) = 0$, $\lim_{\upsilon\to\infty}\alpha(\upsilon) = \infty$, and $\lim_{\upsilon\to\infty}\alpha(\upsilon)/\upsilon = 0$. Since all matches are bilateral and random, a dealer contacts an investor with Poisson rate $\alpha(\upsilon)/\upsilon$. A large measure of dealers can choose to participate in the market, and while they participate, incur a flow cost $\gamma > 0$ that represents the ongoing costs of running the dealership.[20] A steady-state equilibrium with free entry is a list $\langle\{a_i\}, q, \{\phi_i(a)\}, \{n_{ji}\}, \upsilon\rangle$ that satisfies (18)–(21) with $\alpha = \alpha(\upsilon)$ and the free-entry condition $\frac{\alpha(\upsilon)}{\upsilon}\Phi = \gamma$.

For any $\eta > 0$, there exists a steady-state equilibrium with entry of dealers (see Lagos and Rocheteau (2006)). However, the steady-state equilibrium with free entry need not be unique. Although the measure of dealers, υ, is strictly increasing in Φ, the dealers' expected revenue, Φ, can be a nonmonotonic function of $\alpha(\upsilon)$ (part (ii) of Proposition 4). For the case of patient traders, it can be shown that the model necessarily exhibits multiple steady-state equilibria if $\alpha(\upsilon)/\upsilon$ is not too elastic (the effect of an additional dealer on existing dealers' order flow is not too large) and γ is in an intermediate range.[21] In the case of multiple equilibria, the market could operate in a "low-liquidity equilibrium" with small trade volume, large spreads, and long trading delays, merely because few dealers make markets and investors engage in small transactions.[22]

APPENDIX A: PROOFS

PROOF OF LEMMA 1: We can write (5) as

$$V_i(a,t) = \bar{U}_i(a) + \mathbb{E}_i\Big[e^{-r(T_\kappa - t)}\Big\{p(T_\kappa)a + \max_{a'\geq 0}\big[V_{k(T_\kappa)}(a', T_\kappa) - p(T_\kappa)a'\big]\Big\}\Big], \tag{25}$$

[20] This formulation of the free entry of dealers is analogous to the free entry of firms in Pissarides (2000).

[21] See Lagos and Rocheteau (2008) for details.

[22] The strategic complementarity that leads to multiple equilibria in this model depends crucially on the endogenous distribution of asset holdings. The multiplicity is not due to increasing returns in the meeting technology, as in Diamond (1982) or Vayanos and Weill (2008), or to the cost of holding the asset, as in Rocheteau and Wright (2005).

where

$$(26)\qquad \bar{U}_i(a) = \mathbb{E}_i\left[\int_0^{T_\kappa - t} e^{-rs} u_{k(t+s)}(a)\,ds\right].$$

From (25), the problem of an investor with preference shock i who gains access to the market at time t is given by

$$(27)\qquad \max_{a'\geq 0}\left[\bar{U}_i(a') - \left\{p(t) - \mathbb{E}\left[e^{-r(T_\kappa - t)} p(T_\kappa)\right]\right\}a'\right].$$

Equation (26) can be written recursively,

$$(28)\qquad (r+\kappa)\bar{U}_i(a) = u_i(a) + \delta \sum_{j=1}^{I} \pi_j[\bar{U}_j(a) - \bar{U}_i(a)].$$

Multiply (28) through by π_i, sum over i, solve for $\sum_{j=1}^{I} \pi_j \bar{U}_j(a)$, and substitute this expression back into (28) to obtain $\bar{U}_i(a) = \bar{u}_i(a)/(r+\kappa)$, where $\bar{u}_i(a)$ is as in (7). The expected discounted price of the asset at the next time when the investor gets an opportunity to trade is

$$(29)\qquad \mathbb{E}\left[e^{-r(T_\kappa - t)} p(T_\kappa)\right] = \kappa \int_0^\infty e^{-(r+\kappa)s} p(t+s)\,ds.$$

Substitute $\bar{U}_i(a) = \bar{u}_i(a)/(r+\kappa)$ and (29) into (27), and multiply through by $(r+\kappa)$ to see that the investor's problem is given by the maximization in (6). The objective function on the right side of (6) is strictly concave and differentiable, so $u_i'[a_i(t)] - q(t) \leq 0$ ("=" if $a_i(t) > 0$) is necessary and sufficient for an optimum of this problem. Since $q(t) > \bar{u}_i'(\infty)$, $a_i(t)$ given by (6) is the unique solution to the maximization problem on the right side of (6). *Q.E.D.*

PROOF OF LEMMA 2: Since $q(t)$ is continuous and bounded, the right side of (14) is well defined. Rewrite (8) as

$$(30)\qquad q(t) = (r+\kappa)\left[p(t) - \kappa \int_t^\infty e^{-(r+\kappa)(s-t)} p(s)\,ds\right].$$

It can be checked that $\tilde{p}(t) = \frac{1}{r+\kappa}[q(t) + \kappa \int_t^\infty e^{-r(s-t)} q(s)\,ds]$ is a particular solution to (30). Since $q(t)$ is continuous and bounded, $\tilde{p}(t)$ is well defined, and it is continuous and bounded. Suppose that $p(t)$ is any other solution to (30). Then $p(t) = \tilde{p}(t) + z(t)$, where

$$(31)\qquad z(t) = \kappa \int_t^\infty e^{-(r+\kappa)(s-t)} z(s)\,ds.$$

The right side of (31) is differentiable with respect to t, so $z(t)$ is differentiable with $rz(t) - \dot{z}(t) = 0$, which implies $z(t) = Ze^{rt}$ for $Z \in \mathbb{R}$. Hence, any solution $p(t)$ to (30) takes the form

$$p(t) = \frac{1}{r+\kappa}\left[q(t) + \kappa \int_t^\infty e^{-r(s-t)} q(s)\, ds\right] + Ze^{rt}. \tag{32}$$

To determine the value of Z, we use (10), which can be written as

$$\lim_{n\to\infty} \mathbb{E}\left\{e^{-rT_n} p(T_n) \mathbb{E}_i\left[a_{k(T_n)}(T_n)|T_n\right]\right\} = 0. \tag{33}$$

Since (33) holds for each i, multiply the left side by $n_i(0)$ and sum over i to get

$$\lim_{n\to\infty} \mathbb{E}\left\{e^{-rT_n} p(T_n) \sum_{i=1}^I n_i(0)\mathbb{E}_i\left[a_{k(T_n)}(T_n)|T_n\right]\right\} = 0, \tag{34}$$

where

$$\begin{aligned}
&\sum_{i=1}^I n_i(0)\mathbb{E}_i\left[a_{k(T_n)}(T_n)|T_n\right] \\
&\quad = \sum_{i=1}^I n_i(0) \sum_{j=1}^I a_j(T_n) \Pr[k(T_n) = j | k(t) = i] \\
&\quad = \sum_{j=1}^I a_j(T_n) n_j(T_n) \\
&\quad = A.
\end{aligned}$$

Therefore, (34) becomes $\lim_{n\to\infty} \mathbb{E}[e^{-rT_n} p(T_n) A] = 0$, and since $A > 0$, it implies

$$\lim_{n\to\infty} \mathbb{E}[e^{-rT_n} p(T_n)] = 0. \tag{35}$$

Substitute (32) into (35) to obtain

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-rT_n} \frac{q(T_n)}{r+\kappa} + \frac{\kappa}{r+\kappa} \int_{T_n}^\infty e^{-rs} q(s)\, ds\right] + Z = 0. \tag{36}$$

Let F_n denote the distribution function of T_n. Normalize $T_0 = 0$ and notice that $T_n = \sum_{m=1}^n (T_m - T_{m-1})$ is the sum of n independent exponentially distributed random variables with mean $1/\kappa$, so $T_n/n \to 1/\kappa$ almost surely as $n \to \infty$, by the strong law of large numbers. This implies that $\lim_{n\to\infty} F_n(x) = F(x)$ with $F(x) = 0$ for all $x \in [0, \infty)$. Therefore, by Theorem 1 in Feller (1971, p. 249),

$$\lim_{n\to\infty} \int \left[e^{-rx} \frac{q(x)}{r+\kappa} + \frac{\kappa}{r+\kappa} \int_x^\infty e^{-rs} q(s)\, ds \right] dF_n(x)$$
$$= \int \left[e^{-rx} \frac{q(x)}{r+\kappa} + \frac{\kappa}{r+\kappa} \int_x^\infty e^{-rs} q(s)\, ds \right] dF(x)$$
$$= 0$$

since $\lim_{x\to\infty} [e^{-rx} \frac{q(x)}{r+\kappa} + \frac{\kappa}{r+\kappa} \int_x^\infty e^{-rs} q(s)\, ds]$. Hence, (36) implies $Z = 0$. *Q.E.D.*

PROOF OF LEMMA 3: We proceed in three steps: (i) derive $n_{ji}(\tau, t)$, (ii) derive $n^0_{ji}(\mathcal{A}, t)$, and (iii) obtain $H_t(\mathcal{A}, \mathcal{I})$ for an arbitrary $(\mathcal{A}, \mathcal{I}) \in \Sigma$.

(i) The density measure of investors who last readjusted their asset holdings at time $t - \tau > 0$ is $\alpha e^{-\alpha\tau}$. The probability that an investor who last contacted a dealer at time $t - \tau$ has a history of preference types involving $k(t - \tau) = j$ and $k(t) = i$ is $(1 - e^{-\delta\tau})\pi_i + \mathbb{I}_{\{i=j\}} e^{-\delta\tau}$. Since the measure of investors with preference type j at time $t - \tau$ is $n_j(t - \tau)$, and the Poisson process for meeting dealers and the compound Poisson process for preference shocks are independent, the density measure of investors who last traded at time $t - \tau$ and who have a history of preferences involving $k(t - \tau) = j$ and $k(t) = i$ is $n_{ji}(\tau, t) = \alpha e^{-\alpha\tau}[(1 - e^{-\delta\tau})\pi_i + \mathbb{I}_{\{i=j\}} e^{-\delta\tau}] n_j(t - \tau)$, as given by (17).

(ii) The measure of investors who have not contacted a dealer up to time t is $e^{-\alpha t}$. Since the Poisson meeting process is independent of investors' individual states, the time-t measure of investors whose asset holdings and preference types lay in the set $(\mathcal{A}, \{j\})$ at time 0 and who have not yet met a dealer at time t is $e^{-\alpha t} H_0(\mathcal{A}, \{j\})$. The measure of investors who were of preference type j at time 0 and are of type i at time t is $(1 - e^{-\delta t})\pi_i + e^{-\delta t}\mathbb{I}_{\{j=i\}}$. Thus, the time-$t$ measure of investors who at time 0 had preference type j and assets in $\mathcal{A}$, whose preference type is i at the current time t, and who have never traded (so their asset holdings are still in $\mathcal{A}$) is $n^0_{ji}(\mathcal{A}, t) = e^{-\alpha t}[(1 - e^{-\delta t})\pi_i + e^{-\delta t}\mathbb{I}_{\{j=i\}}] H_0(\mathcal{A}, \{j\})$, as given in (16).

(iii) $H_t(\mathcal{A}, \mathcal{I})$ is the measure of investors who have an individual state $(a, i) \in (\mathcal{A}, \mathcal{I})$ at time t. The first term in $H_t(\mathcal{A}, \mathcal{I})$ is $\sum_{i\in\mathcal{I}} \sum_{j=1}^{I} n^0_{ji}(\mathcal{A}, t)$, namely, those investors who never contacted dealers but who were holding asset positions in the set $\mathcal{A}$ at time 0 and whose preference types at t lie in $\mathcal{I}$. The time-t measure of investors of type i who chose an asset position in the set $\mathcal{A}$ the last time they traded, given that their preference type at that time was j, is $\int_0^t \mathbb{I}_{\{a_j(t-\tau)\in\mathcal{A}\}} n_{ji}(\tau, t)\, d\tau$. Thus, the second term in $H_t(\mathcal{A}, \mathcal{I})$, namely, the measure of investors who the last time they traded chose asset positions that belong to the set $\mathcal{A}$ and whose preference types at time t lie in $\mathcal{I}$, is $\sum_{i\in\mathcal{I}} \sum_{j=1}^{I} \int_0^t \mathbb{I}_{\{a_j(t-\tau)\in\mathcal{A}\}} n_{ji}(\tau, t)\, d\tau$. *Q.E.D.*

PROOF OF PROPOSITION 1: For all $t \geq 0$, the distribution $\{n_i(t)\}_{i=1}^I$ is unique and given by (12). Define $A_t^d(q) \equiv \{\sum_{i=1}^I n_i(t)a_i(q) : a_i(q) \in \arg\max_{a' \geq 0}[\bar{u}_i(a') - qa']\}$ for $q \in (\underline{q}(t), +\infty)$, where $\underline{q}(t) = \max_{i \in \mathbb{X}} \bar{u}_i'(\infty) \times \mathbb{I}_{\{n_i(t)>0\}}$. (If $q \leq \underline{q}(t)$, then (9) has no solution for some i such that $n_i(t) > 0$.) From Lemma 1, the optimal choice a_i is uniquely determined for all $q \in (\underline{q}(t), +\infty)$ and all i such that $n_i(t) > 0$, and it is continuous in q. Consequently, $A_t^d(q)$ is single-valued and continuous for $q \in (\underline{q}(t), +\infty)$. Moreover, (9) implies that any interior choice $a_i(t)$ is a strictly decreasing function of $q(t)$ for every i. Thus, $A_t^d(q)$ is strictly decreasing for all $q \in (\underline{q}(t), \bar{q}(t))$, where $\bar{q}(t) = \max_{i \in \mathbb{X}} \bar{u}_i'(0)\mathbb{I}_{\{n_i(t)>0\}}$ and $A_t^d(q) = \{0\}$ for all $q \geq \bar{q}(t)$. As $q \downarrow \underline{q}(t)$, $A_t^d(q) \to +\infty$, and as $q \uparrow \bar{q}(t)$, $A_t^d(q) \to 0$. So for each t there is a unique $q(t) \in (\underline{q}(t), \bar{q}(t))$ such that $A_t^d[q(t)] = \{A\}$ or, equivalently, such that $\sum_{i=1}^I n_i(t)a_i[q(t)] = A$. Given this $q(t)$, there is a unique $\{a_i(t)\}_{i=1}^I$ that solves (9). Given $q(t)$, (11) gives the fee $\phi_i(a,t)$ for every i and a. Finally, given $\{a_i(t)\}_{i=1}^I$, the distribution H_t is given by (15).

From (12), $\lim_{t\to\infty} n_i(t) = \pi_i$ for each i. By an argument similar to that in the proof of Proposition 1, one can establish that there is a unique, time-invariant q that clears the asset market. Given this q, (9) implies a unique set of time-invariant optimal asset holdings $\{a_i\}_{i=1}^I$. Thus, $\{a_i\}_{i=1}^I$ and q satisfy (18) and (19). Given the fact that $q(t) = q$ for all t, (14) implies $p = q/r$. Given q and $\{a_i\}_{i=1}^I$, (11) implies (20), which determines the time-invariant fees $\{\phi_i(a)\}_{i=1}^I$. To derive (21), start from Lemma 3 and note that $\lim_{t\to\infty} n_{ji}^0(\mathcal{A}, t) = 0$ for all $i, j \in \mathbb{X}$ and all $\mathcal{A} \subseteq \mathbb{R}_+$. Also, $\lim_{t\to\infty} n_{ji}(\tau, t) = \alpha e^{-\alpha\tau}[(1 - e^{-\delta\tau})\pi_i + e^{-\delta\tau}\mathbb{I}_{\{i=j\}}]\pi_j \equiv n_{ji}(\tau, \infty)$ and $\lim_{t\to\infty} a_j(t-\tau) = a_j$, so

$$\lim_{t\to\infty} H_t(\mathcal{A}, \mathcal{I}) = \sum_{i\in\mathcal{I}} \sum_{j=1}^I \left[\int_0^\infty \mathbb{I}_{\{a_j \in \mathcal{A}\}} n_{ji}(\tau, \infty)\, d\tau\right] \equiv H(\mathcal{A}, \mathcal{I})$$

for all $(\mathcal{A}, \mathcal{I}) \in \Sigma$. To conclude, observe that $H(\{a_i\}, \{j\}) = \int_0^\infty n_{ij}(\tau, \infty)\, d\tau$ and carry out the integration to obtain (21). *Q.E.D.*

PROOF OF PROPOSITION 2: (i) From (7), as $r + \kappa \to 0$, then $\bar{u}_i(a) \to \delta \sum_{j=1}^I \pi_j u_j(a)$ which is independent of i. Together with market clearing, this implies that $a_i \to A$ for all $i \in \{1, \ldots, I\}$ as $r + \kappa \to 0$.

(ii) Let $a_i(\kappa)$ denote the individual demand of an investor with preference type i in a market with effective contact rate κ. With $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$,

$$a_i(\kappa) = A\Bigg/\left(\sum_{j=1}^I \pi_j \left[\frac{(r+\kappa)\varepsilon_j + \delta\bar{\varepsilon}}{(r+\kappa)\varepsilon_i + \delta\bar{\varepsilon}}\right]^{1/\sigma}\right). \tag{37}$$

Consider $\kappa' > \kappa$. We have $a_1(\kappa') < a_1(\kappa)$, since

$$\frac{(r+\kappa')\varepsilon_j + \delta\bar{\varepsilon}}{(r+\kappa')\varepsilon_1 + \delta\bar{\varepsilon}} > \frac{(r+\kappa)\varepsilon_j + \delta\bar{\varepsilon}}{(r+\kappa)\varepsilon_1 + \delta\bar{\varepsilon}} \quad \text{for all } j > 1,$$

and $a_I(\kappa') > a_I(\kappa)$, since

$$\frac{(r+\kappa')\varepsilon_j + \delta\bar{\varepsilon}}{(r+\kappa')\varepsilon_I + \delta\bar{\varepsilon}} < \frac{(r+\kappa)\varepsilon_j + \delta\bar{\varepsilon}}{(r+\kappa)\varepsilon_I + \delta\bar{\varepsilon}} \quad \text{for all } j < I.$$

The difference $a_i(\kappa') - a_i(\kappa)$ is continuous in ε_i, so there exists $\tilde{\varepsilon} \in (\varepsilon_1, \varepsilon_I)$ such that $a_i(\kappa') = a_i(\kappa) \equiv \tilde{a}$. Moreover, from (37),

$$\left.\frac{\partial a_i(\kappa')}{\partial \varepsilon_i}\right|_{\varepsilon_i=\tilde{\varepsilon}} = \frac{(r+\kappa')\tilde{a}}{\sigma[(r+\kappa')\tilde{\varepsilon} + \delta\bar{\varepsilon}]} > \frac{(r+\kappa)\tilde{a}}{\sigma[(r+\kappa)\tilde{\varepsilon} + \delta\bar{\varepsilon}]} = \left.\frac{\partial a_i(\kappa)}{\partial \varepsilon_i}\right|_{\varepsilon_i=\tilde{\varepsilon}},$$

so $a_i(\kappa')$ as a function of ε_i intersects $a_i(\kappa)$ from below. Hence $\tilde{\varepsilon}$ is unique, and $a_i(\kappa') < a_i(\kappa)$ for all $\varepsilon_i < \tilde{\varepsilon}$ and $a_i(\kappa') > a_i(\kappa)$ for all $\varepsilon_i > \tilde{\varepsilon}$. With (21), the cumulative distribution of assets indexed by κ, is

$$\mathbb{G}_\kappa(a) = \sum_{j=1}^{I} \mathbb{I}_{\{a_j(\kappa) \le a\}} \pi_j.$$

The fact that $a_i(\kappa') < a_i(\kappa)$ for $\varepsilon_i < \tilde{\varepsilon}$ implies $\mathbb{G}_{\kappa'}(a) \ge \mathbb{G}_\kappa(a)$ for all $a < \tilde{a}$. Thus, $\int_0^a \mathbb{G}_{\kappa'}(x)\,dx \ge \int_0^a \mathbb{G}_\kappa(x)\,dx$ for all $a < \tilde{a}$. Moreover,

$$\int_0^{a_I(\kappa')} \mathbb{G}_{\kappa'}(x)\,dx = \int_0^{a_I(\kappa')} \mathbb{G}_\kappa(x)\,dx,$$

so

$$\int_0^a \mathbb{G}_{\kappa'}(x)\,dx - \int_0^a \mathbb{G}_\kappa(x)\,dx = \int_a^\infty \mathbb{G}_\kappa(x)\,dx - \int_a^\infty \mathbb{G}_{\kappa'}(x)\,dx.$$

The right side of this expression is nonnegative for $a \ge \tilde{a}$ because $a_i(\kappa') \ge a_i(\kappa)$ for $\varepsilon_i \ge \tilde{\varepsilon}$. Thus,

$$\int_0^a \mathbb{G}_{\kappa'}(x)\,dx \ge \int_0^a \mathbb{G}_\kappa(x)\,dx \quad \text{for all } a \ge \tilde{a}.$$

We conclude that

$$\int_0^a \mathbb{G}_{\kappa'}(x)\,dx \ge \int_0^a \mathbb{G}_\kappa(x)\,dx \quad \text{for all } a \ge 0,$$

that is, $\mathbb{G}_\kappa$ second-order stochastically dominates $\mathbb{G}_{\kappa'}$. *Q.E.D.*

PROOF OF PROPOSITION 3: (i) Part (i) follows immediately from (24) and part (ii) of Proposition 2.

(ii) Since $u_i(a) = \varepsilon_i \ln a$, we have $a_i > 0$ for all i and $a_i \neq a_j$ unless $i = j$. From (21), the proportion of trades that involve buying a_i and selling a_j or vice versa (for $i \neq j$) is $(n_{ij} + n_{ji})/(1 - \sum_{i=1}^{I} n_{ii}) = 2\pi_i\pi_j/(1 - \sum_{i=1}^{I} \pi_i^2)$, which is independent of κ. From Proposition 5, $dp/d\kappa = 0$, so differentiating (22),

$$\frac{d[g_i(\kappa; p) - g_j(\kappa; p)]}{d\kappa} = \frac{\delta(\varepsilon_i - \varepsilon_j)}{rp(r + \kappa + \delta)^2}.$$

Thus, $|a_i - a_j| = |g_i(\kappa; p) - g_j(\kappa; p)|$ increases with κ for all $i \neq j$. The measure of trades of size less than $z \geq 0$ is

$$\sum_{i=1}^{I} \sum_{j \neq i} \frac{\pi_i \pi_j}{1 - \sum_{i=1}^{I} \pi_i^2} \mathbb{I}_{\{|a_i - a_j| \leq z\}},$$

which is decreasing in κ. This establishes that the distribution of trade sizes associated with κ' first-order stochastically dominates the one associated with κ if $\kappa' > \kappa$. Since every trade size is larger in the market with a larger κ, we conclude that $\mathcal{V}$ increases with κ.

(iii) For $I = 2$, we have $\mathbb{X} = \{1, 2\}$ and

$$\mathcal{V} = \frac{\alpha\delta\pi_1\pi_2}{\alpha + \delta}[a_2(\kappa) - a_1(\kappa)],$$

where $a_i(\kappa)$ is given by (37). Since $\varepsilon_1 < \varepsilon_2$, we have $a_1(\kappa) < a_2(\kappa)$, and by part (i) of Proposition 2,

$$\frac{da_1(\kappa)}{d\kappa} < 0 < \frac{da_2(\kappa)}{d\kappa}.$$

To find $\frac{d\mathcal{V}}{d\kappa}$, we consider two cases: (a) An increase in κ caused by a decrease in η (keeping α constant). For this case,

$$\frac{d\mathcal{V}}{d\kappa} = \frac{\alpha\delta\pi_1\pi_2}{\alpha + \delta}\left[\frac{da_2(\kappa)}{d\kappa} - \frac{da_1(\kappa)}{d\kappa}\right] > 0.$$

(b) An increase in κ caused by an increase in α, which implies

$$\frac{d\mathcal{V}}{d\kappa} = \left(\frac{\delta}{\alpha + \delta}\right)^2 \pi_1\pi_2[a_2(\kappa) - a_1(\kappa)] + \frac{\alpha\delta\pi_1\pi_2}{\alpha + \delta}\left[\frac{da_2(\kappa)}{d\kappa} - \frac{da_1(\kappa)}{d\kappa}\right] > 0.$$

Q.E.D.

PROOF OF LEMMA 4: Differentiate (20) to obtain

$$\frac{\partial \phi_i(a)}{\partial a} = -\frac{\eta}{r+\kappa}[\bar{u}_i'(a) - q].$$

Suppose that the nonnegativity constraint on a_i is slack. Then, since $\bar{u}_i$ is strictly concave and $\bar{u}_i'(a_i) - q = 0$, we know that $\bar{u}_i'(a) - q < 0$ if and only if $a - a_i > 0$, and $\partial \phi_i(a)/\partial a$ has the same sign as $a - a_i$. If $a_i = 0$, then $a > a_i$ and $\bar{u}_i'(a) - q < \bar{u}_i'(a_i) - q \leq 0$, so $\partial \phi_i(a)/\partial a > 0$, which is the same sign as $a - a_i = a > 0$. This establishes the first part. To show the second part, divide (20) by $(a_i - a)$ and differentiate the resulting expression to get

$$\frac{\partial}{\partial a}\left[\frac{\phi_i(a)}{a_i - a}\right] = \frac{\eta}{r+\kappa}\left[\frac{\bar{u}_i(a_i) - \bar{u}_i(a) - \bar{u}_i'(a)(a_i - a)}{(a_i - a)^2}\right],$$

which is strictly negative, since $\bar{u}_i$ is strictly concave. *Q.E.D.*

PROOF OF PROPOSITION 4: (i) Let $q(\kappa, r)$, $a_i(\kappa, r)$, and $\phi_{ji}(\kappa, r)$ denote, respectively, the equilibrium q, a_i, and ϕ_{ji} that solve (18), (19), and (20) for all $i, j \in \mathbb{X}$. We proceed in three steps: (a) show that $\phi_{ji}(\kappa, r) > 0$ for all $\kappa \in (0, \infty)$ and all $r \in [0, \infty)$ provided $a_i(\kappa, r) \neq a_j(\kappa, r)$ and $\eta > 0$; (b) establish that $\lim_{\kappa \to \infty} \phi_{ji}(\kappa, r) = 0$ for any $r \geq 0$ and all $(i, j) \in \mathbb{X}^2$; (c) show that for each $\kappa \in (0, \infty)$ there is $\bar{r} > 0$ such that $\phi_{ji}(0, r) < \phi_{ji}(\kappa, r)$ for all $r \in (0, \bar{r})$. The nonmonotonicity of $\phi_{ji}(\kappa, r)$ with respect to κ for all $r \in [0, \bar{r})$ will then follow from steps (a) through (c).

(a) From (20), $\phi_{ij} = \frac{\eta}{r+\kappa}\{\max_{a'}[\bar{u}_i(a'; \kappa, r) - qa'] - [\bar{u}_i(a_j; \kappa, r) - qa_j]\}$, so $\phi_{ij}(\kappa, r) > 0$ for all $\kappa \in (0, \infty)$ and all $r \in [0, \infty)$ provided $\eta > 0$ and $a_j \neq \arg\max_{a' \geq 0}[\bar{u}_i(a') - qa']$ (i.e., provided the investor trades).

(b) $\lim_{\kappa \to \infty} q(\kappa, r) = q^*$ and $\lim_{\kappa \to \infty} a_i(\kappa, r) = \arg\max_{a' \geq 0}[u_i(a') - q^* a'] \equiv h_i^\infty(q^*)$, where q^* is independent of r and solves $\sum_{i=1}^I \pi_i h_i^\infty(q^*) = A$, which in turn implies $q^* \in (0, \infty)$, $h_i^\infty(q^*) < \infty$, and hence $|u_i(a_j) - q^* a_j| < \infty$ for all $(i, j) \in \mathbb{X}^2$. Therefore, $\lim_{\kappa \to \infty} \phi_{ij}(\kappa, r) = 0$ for any $r \geq 0$ and all $(i, j) \in \mathbb{X}^2$.

(c) Let $\kappa \to 0$ to obtain $q(0, r) = \tilde{q}(r)$ and $a_i(0, r) = \arg\max_{a' \geq 0}[\tilde{u}_i(a') - \tilde{q}a'] \equiv h_i^0(\tilde{q})$, where $\tilde{u}_i(a; r) = (ru_i(a) + \delta\tilde{u}(a))/(r + \delta)$, $\tilde{u}(a) = \sum_{k=1}^I \pi_k u_k(a)$, and $\tilde{q}$ solves $\sum_{i=1}^I \pi_i h_i^0(\tilde{q}) = A$. From (20),

$$\phi_{ji}(0, r) = \eta\left(\left\{r[u_i(a_i) - u_i(a_j)] + \delta\sum_{k=1}^I \pi_k[u_k(a_i) - u_k(a_j)]\right\} - (r + \delta)\tilde{q}(r)(a_i - a_j)\right) \Big/ ((r + \delta)r). \tag{38}$$

Observe that $\lim_{r\to 0} a_i(0,r) = \tilde{u}'^{-1}[\tilde{q}(0)] = A$ for each $i \in \mathbb{X}$. Totally differentiate (18) and (19) with respect to r and evaluate at $\kappa = r = 0$ to find

$$\frac{\partial a_i(0,0)}{\partial r} = \frac{\tilde{q}(0) + \delta \tilde{q}'(0) - u_i'(A)}{\delta \sum_{k=1}^{I} \pi_k u_k''(A)}$$

and

$$\sum_{i=1}^{I} \pi_i \frac{\partial a_i(0,0)}{\partial r} = 0.$$

Combine these conditions to get

$$\frac{\tilde{q}(0) + \delta \tilde{q}'(0) - \tilde{u}'(A)}{\delta \sum_{k=1}^{I} \pi_k u_k''(A)} = 0,$$

which together with the investor's first-order condition, $\tilde{u}'(A) = \tilde{q}(0)$, implies $\tilde{q}'(0) = 0$ and hence

$$\frac{\partial a_i(0,0)}{\partial r} = \frac{\tilde{q}(0) - u_i'(A)}{\delta \sum_{k=1}^{I} \pi_k u_k''(A)}.$$

With this, apply l'Hôpital's rule to (38) to find $\lim_{r\to 0} \phi_{ji}(0,r) = 0$.

Our assumptions on primitives imply that $q(\kappa, r)$ and $a_i(\kappa, r)$ are continuous functions, so $\phi_{ji}(\kappa, r)$ is continuous. Hence, for each (i, j) with $i \neq j$ and each $\kappa \in (0, \infty)$, there is some $\bar{r} > 0$ such that for all $r \in [0, \bar{r})$, we have $\lim_{\kappa\to\infty} \phi_{ji}(\kappa, r) = 0 < \phi_{ji}(\kappa, r)$ (by (a) and (b)) and $\phi_{ji}(0, r) < \phi_{ji}(\kappa, r)$ (by (a) and (c)), which establishes the nonmonotonicity of ϕ_{ij} with respect to κ.

(ii) Write $\Phi(\alpha, \eta, r) = \sum_{i,j=1}^{I} n_{ji}(\alpha)\phi_{ji}[\alpha(1-\eta), r]$, where $n_{ji}(\alpha)$ is given by (21). Fix an arbitrary $(\alpha, \eta) \in (0, \infty) \times (0, 1)$. From step (a) in part (i), $\phi_{Ij}[\alpha(1-\eta), r] > 0$ for $j < I$ and all $r \in [0, \infty)$. Hence, $\Phi(\alpha, \eta, r) > 0$ for all $\alpha(1-\eta) \in (0, \infty)$ and all $r \in [0, \infty)$. Following a similar reasoning as in step (c) in part (i), for each $(i, j) \in \mathbb{X}^2$, there is $\bar{r}_{ji} > 0$ such that for all $r \in [0, \bar{r}_{ji})$, $\phi_{ji}(0, r) < \Phi(\alpha, \eta, r)$. Then $\Phi(0, \eta, r) < \Phi(\alpha, \eta, r)$ for any $r \in [0, r_0)$, where $r_0 = \min_{(i,j)\in\mathbb{X}^2} \bar{r}_{ji}$. Finally, from step (b) in part (i), for any $r \geq 0$ we have $\lim_{\alpha'\to\infty} \Phi(\alpha', \eta, r) = 0 < \Phi(\alpha, \eta, r)$, which establishes the nonmonotonicity of Φ with respect to α and, therefore, with respect to $\kappa = \alpha(1-\eta)$. *Q.E.D.*

PROPOSITION 5: *Let $u_i(a) = \varepsilon_i u(a)$. If $[u'(a)]^2/u''(a)$ is strictly decreasing in a, then $dp/d\kappa > 0$. If $[u'(a)]^2/u''(a)$ is increasing in a, then $dp/d\kappa \leq 0$ (with "$=$" if $[u'(a)]^2/u''(a)$ is constant).*

PROOF: Differentiate (19) to obtain

$$\frac{dp}{d\kappa} = \frac{\sum_{i=1}^{I} \pi_i \, \partial a_i/\partial \kappa}{-\sum_{i=1}^{I} \pi_i \, \partial a_i/\partial p}.$$

The denominator of this expression is strictly positive (from (22)), so focus on the sign of the numerator. Differentiate (22) to obtain $\partial a_i/\partial \kappa$, multiply by π_i, and add over all i to arrive at

$$\sum_{i=1}^{I} \pi_i \frac{\partial a_i}{\partial \kappa} = \frac{\delta}{(r+\kappa+\delta)^2 rp} \sum_{i=1}^{I} \pi_i \frac{[u'(a_i)]^2}{-u''(a_i)} (\varepsilon_i - \bar{\varepsilon}).$$

Suppose $-[u'(a)]^2/u''(a)$ is strictly increasing in a. Let $\bar{a}$ denote the a that solves (22) for $\bar{\varepsilon}_i = \bar{\varepsilon}$. Then note that $-[u'(a_i)]^2(\varepsilon_i - \bar{\varepsilon})/u''(a_i) \geq -[u'(\bar{a})]^2(\varepsilon_i - \bar{\varepsilon})/u''(\bar{a})$ for each i, with strict inequality for all i such that $\varepsilon_i \neq \bar{\varepsilon}$. Thus, $\sum_{i=1}^{I} \pi_i \, \partial a_i/\partial \kappa > 0$ and consequently, $\frac{dp}{d\kappa} > 0$. Similar reasoning implies $\frac{dp}{d\kappa} < 0$ if $-[u'(a)]^2/u''(a)$ is strictly decreasing and $\frac{dp}{d\kappa} = 0$ if $-[u'(a)]^2/u''(a)$ is constant in a. *Q.E.D.*

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Econometrica Supplementary Material

SUPPLEMENT TO "LIQUIDITY IN ASSET MARKETS WITH SEARCH FRICTIONS"
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BY RICARDO LAGOS AND GUILLAUME ROCHETEAU

APPENDIX B: GENERALIZED PREFERENCE SHOCKS

IN THIS APPENDIX we generalize the stochastic process for preference shocks as follows. As before, each investor receives a preference shock with Poisson arrival rate δ, and this process is independent across investors. But now we let $\Pi = [\pi_{ij}]$ denote an $I \times I$ matrix and assume that conditional on receiving a preference shock, an investor with preference type i draws preference type j with probability $\pi_{ij} > 0$, with $\sum_{j=1}^{I} \pi_{ij} = 1$ for all $i \in \mathbb{X}$. The formulation studied in the body of the paper corresponds to the i.i.d. case, $\pi_{ij} = \pi_j$ for all i.

Equilibrium

The investor's value function $V_i(a, t)$ still satisfies (1) and the dealer's value function is unchanged. The bargaining outcome is also unchanged, so $V_i(a, t)$ also satisfies (5). The following lemma generalizes Lemma 1.

LEMMA 5: *An investor with preference type i and asset holdings a who readjusts his asset position at time t solves*

$$\max_{a' \geq 0}[\bar{u}_i(a') - q(t)a'], \tag{39}$$

where

$$\bar{u}_i(a) = \sum_{k=0}^{\infty} \sum_{j=1}^{I} \mu_k \pi_{ij}^{(k)} u_j(a) \quad \textit{for} \quad i = 1, \ldots, I, \tag{40}$$

$$q(t) = (r + \kappa)\left[p(t) - \kappa \int_0^{\infty} e^{-(r+\kappa)s} p(t + s)\, ds\right], \tag{41}$$

$\Pi^k = [\pi_{ij}^{(k)}]$ *for* $k \geq 1$, $\pi_{ij}^{(0)} = \mathbb{I}_{\{j=i\}}$, *and* $\mu_k = (\frac{r+\kappa}{r+\kappa+\delta})(\frac{\delta}{r+\kappa+\delta})^k$.

PROOF: As before, $V_i(a, t)$ satisfies (25), so the problem of an investor with preference shock i who gains access to the market at time t is given by (27) with $\bar{U}_i(a)$ as in (26). Notice that (29) is unchanged, so we only have to calculate $\bar{U}_i(a)$. Equation (26) can be written as

$$(r + \kappa + \delta)\bar{U}_i(a) = u_i(a) + \delta \sum_{j=1}^{I} \pi_{ij} \bar{U}_j(a) \quad \text{for} \quad i = 1, \ldots, I$$

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or, equivalently,

$$\left(\mathbf{I} - \frac{\delta}{r+\kappa+\delta}\Pi\right)\bar{\mathbf{u}} = \frac{r+\kappa}{r+\kappa+\delta}\mathbf{u}, \tag{42}$$

where $\mathbf{I}$ is the $I \times I$ identity matrix, and $\bar{\mathbf{u}}$ and $\mathbf{u}$ are $I \times 1$ vectors with ith entry $\bar{u}_i(a) \equiv (r+\kappa)\bar{U}_i(a)$ and $u_i(a)$, respectively. Since $\lim_{k\to\infty}(\frac{\delta}{r+\kappa+\delta}\Pi)^k = 0$, $(\mathbf{I}-\frac{\delta}{r+\kappa+\delta}\Pi)^{-1}$ exists, $\sum_{k=0}^{\infty}(\frac{\delta}{r+\kappa+\delta}\Pi)^k$ converges, and $(\mathbf{I}-\frac{\delta}{r+\kappa+\delta}\Pi)^{-1} = \sum_{k=0}^{\infty}(\frac{\delta}{r+\kappa+\delta}\Pi)^k$. Thus

$$\bar{\mathbf{u}} = \sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa+\delta}\Pi\right)^k \frac{r+\kappa}{r+\kappa+\delta}\mathbf{u},$$

which can be written as in (40). Substitute $\bar{U}_i(a) = \bar{u}_i(a)/(r+\kappa)$ and (29) into (27), and multiply through by $(r+\kappa)$ to obtain the formulation of the investor's problem stated in the lemma. *Q.E.D.*

Intuitively, $\bar{u}_i(a)/(r+\kappa)$ is the expected discounted utility to an investor with preference type i from holding a until the next (*effective*) time when he readjusts his holdings. We can write

$$\bar{u}_i(a) = \sum_{k=0}^{\infty}\mu_k \bar{u}_i^{(k)}(a),$$

where μ_k is the probability the investor receives k preference shocks before his next effective contact with a dealer, and

$$\bar{u}_i^{(k)}(a) \equiv \sum_{j=1}^{I}\pi_{ij}^{(k)} u_j(a) \tag{43}$$

is his expected utility conditional on preference type i and conditional on his receiving k preference shocks over that time period. With this generalized expression for $\bar{u}_i(a)$, a choice of asset holdings, $a_i(t)$, still satisfies (9), and Lemma 2 and (11) remain unchanged.

The law of motion for the measure of investors with preference type i is

$$\dot{n}_i(t) = \delta\sum_{j=1}^{I}\pi_{ji}n_j(t) - \delta n_i(t),$$

which implies $\mathbf{n}(t) = \mathbf{n}(0)e^{\delta(\Pi-\mathbf{I})t}$, where $\mathbf{I}$ is the $I \times I$ identity matrix and $\mathbf{n}(t)$ denotes the $1 \times I$ vector with ith element $n_i(t)$. Thus

$$n_i(t) = \sum_{j=1}^{I} \rho_{ji}(t) n_j(0), \tag{44}$$

where $\rho_{ji}(t)$ denotes the jith element of the matrix $e^{\delta(\Pi-\mathbf{I})t}$ and represents the transition probability for an investor from preference type j to preference type i in a period of length t. The clearing condition in the interdealer market is still (13), but with $n_i(t)$ given by (44). With this, it is straightforward to show that Lemma 3 generalizes as follows.

LEMMA 6: *The measure of investors across individual states at time t satisfies* (15) *for all* $(\mathcal{A}, \mathcal{I}) \in \Sigma$, *where*

$$n^0_{ji}(\mathcal{A}, t) = e^{-\alpha t} \rho_{ji}(t) H_0(\mathcal{A}, \{j\}), \tag{45}$$

$$n_{ji}(\tau, t) = \alpha e^{-\alpha \tau} \rho_{ji}(\tau) n_j(t - \tau). \tag{46}$$

An equilibrium is a time path $\langle \{a_i(t)\}, q(t), p(t), \{\phi_i(a, t)\}, H_t \rangle$ that satisfies (9) (with $\bar{u}_i(a)$ given by (40)), (14), (11), (13) (with $n_i(t)$ given by (44)), and (15) (with $n^0_{ji}(\mathcal{A}, t)$ and $n_{ji}(\tau, t)$ given by (45) and (46), respectively). The proof of Proposition 1 can be immediately extended to show that there exists a unique equilibrium. In the limiting case $\alpha \to \infty$, we have $\bar{u}_i(a) \to u_i(a)$ (from (42)) and $u'_i[a_i(t)] \le q(t) = rp(t) - \dot{p}(t)$ for all i (from (8) and (9)). Also, $q(t) \to q^*(t)$, where $q^*(t)$ solves $\sum_{i \in \mathcal{I}^+_t} n_i(t) u'^{-1}_i [q^*(t)] = A$ and $\mathcal{I}^+_t = \{i \in \mathbb{X} : a_i(t) > 0\}$ (from (13)), and $\phi_i(a, t) \to 0$ for all a, i, and t (from (11)). Finally, $\alpha \to \infty$ implies that every investor holds his desired asset position at all times. Thus, as before, the equilibrium fees, asset price, and distribution of asset holdings converge to their Walrasian counterparts as frictions vanish.

Efficiency

The planner's problem is

$$\max_{\{a_i(t)\}} \int_0^\infty \frac{\alpha}{r + \alpha} \sum_{i=1}^{I} \hat{u}_i[a_i(t)] n_i(t) e^{-rt} \, dt$$

subject to $\sum_{i=1}^{I} n_i(t) a_i(t) \le A$, where $n_i(t)$ is given by (44) and $\hat{u}_i(a) = \sum_{k=0}^{\infty} \hat{\mu}_k \bar{u}^{(k)}_i$, with $\hat{\mu}_k = (\frac{r+\alpha}{r+\alpha+\delta})(\frac{\delta}{r+\alpha+\delta})^k$. The first-order necessary and sufficient conditions are (a) $\hat{u}'_i[a_i(t)] \le \lambda(t)$ for $i = 1, \dots, I$ (with "=" if $a_i(t) > 0$), where $\lambda(t)$ is the multiplier on the resource constraint, and (b) $\sum_{i=1}^{I} n_i(t) \times a^*_i[\lambda(t)] = A$, where $a^*_i[\lambda(t)]$ is the $a_i(t)$ that satisfies (a). Notice that $\hat{\mu}_k = \mu_k$, and hence $\hat{u}_i = \bar{u}_i$ if and only if $\eta = 0$. Hence, if we set $q(t) = \lambda(t)$ we find that the competitive allocation $\{a_i(t)\}$ coincides with the efficient allocation $\{a_i^*(t)\}$ if and only if $\eta = 0$.

Steady State

Our assumptions ensure that there exists a unique row vector $\pi^* = [\pi_i^*]$ such that $\pi^*(\Pi - \mathbf{I}) = \mathbf{0}$ with $\sum_{i=1}^I \pi_i^* = 1$ and that $\lim_{t\to\infty} \rho_{ji}(t) = \pi_i^*$. Hence, (44) implies $\lim_{t\to\infty} n_i(t) = \pi_i^*$ for all i. The generalization of the second part of Proposition 1 is straightforward. The equilibrium allocations and prices $\langle \{a_i(t)\}, q(t), p(t), \{\phi_i(a,t)\}, H_t\rangle$ converge to the unique steady-state allocations and prices $\langle \{a_i\}, q, p, \{\phi_i(a)\}, H\rangle$ that satisfy $p = q/r$, $\bar{u}_i'(a_i) \le q$ ("$=$" if $a_i > 0$, with $\bar{u}_i$ as in (40)), $\sum_{i=1}^I \pi_i^* a_i = A$, $\phi_i(a)$ as in (20), and $\lim_{t\to\infty} H_t(\mathcal{A},\mathcal{I}) = H(\mathcal{A},\mathcal{I})$, where $H(\{a_j\},\{i\}) = \pi_j^* \int_0^\infty \alpha e^{-\alpha\tau}\rho_{ji}(\tau)\,d\tau$ and $H(\mathcal{A},\mathcal{I}) = 0$ for all $(\mathcal{A},\mathcal{I}) \in \Sigma$ such that $\bigcup_{j=1}^I \{a_j\} \cap \mathcal{A} = \emptyset$.

Asset Positions, Prices, and Trade Volume

Focus on the steady state and assume $u_i'(0) = \infty$ and $u_i'(\infty) = 0$ for each i. An investor's asset choice satisfies

$$\sum_{k=0}^{\infty} \mu_k \bar{u}_i^{(k)\prime}(a_i) = rp. \tag{47}$$

As before, when an investor with preference type i chooses his asset holdings, he evaluates his expected marginal utility from holding the asset until the next trading time. If he is hit by k preference shocks over the holding period, his expected marginal utility from a_i is $\bar{u}_i^{(k)\prime}(a_i)$. Since the number of preference shocks he experiences is random, the investor also takes expectations over $\bar{u}_i^{(k)\prime}(a_i)$ using the (discounting-adjusted) probability distribution of preference shocks, $\{\mu_k\}_{k=0}^\infty$.

Let $a_i = g_i(\kappa; p)$ denote the choice of asset holdings characterized by (47). Then

$$\frac{\partial g_i(\kappa; p)}{\partial \kappa} = \frac{\sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa} - k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i)}{-\bar{u}_i''(a_i)(r+\kappa+\delta)}, \tag{48}$$

which generalizes (23), has the sign of the numerator. From (47), notice that κ only affects the probability distribution $\{\mu_k\}$; intuitively, a marginal increase in κ increases the probability of k preference shocks for $k < \frac{\delta}{r+\kappa}$ and decreases it for $k > \frac{\delta}{r+\kappa}$. This means that an increase in κ induces the investor to put more weight on $\bar{u}_i^{(k)}$'s with smaller k. If shocks are i.i.d. as in the body of the paper (i.e., $\pi_{ij} = \pi_j$ for all i), then $\bar{u}_i^{(0)\prime}(a_i) = u_i'(a_i)$ and $\bar{u}_i^{(k)\prime}(a_i) = \sum_{j=1}^{I} \pi_j u_j'(a_i)$ for all $k \geq 1$, so in terms of preference shocks over the holding period, there are just two relevant events: either none hits or at least one hits. An increase in κ raises the probability of the former and reduces the probability of the latter, so it makes an investor with preference type i choose a larger asset position if and only if $u_i'(a_i) > \sum_{j=1}^{I} \pi_j u_j'(a_i)$. Analogously, according to (48), in this more general formulation an investor with preference type i increases his asset demand in response to an increase in κ if and only if $u_i'(a_i) > \sum_{k=1}^{\infty}(\frac{\delta}{r+\kappa} - k) \times \mu_{k-1}\bar{u}_i^{(k)\prime}(a_i)$. Since this condition may seem intricate, we provide simpler conditions for some special cases.

PROPOSITION 6: (i) *Suppose the sequence* $\{\bar{u}_i^{(k)\prime}(a_i)\}_{k=0}^{\infty}$ *is monotone in* k. *Then* $\partial g_i(\kappa; p)/\partial\kappa > 0$ *if and only if*

$$u_i'(a_i) > \sum_{j=1}^{I} \pi_j^* u_j'(a_i). \tag{49}$$

(ii) *Consider the frictionless limit*, $\kappa \to \infty$. *Then*

$$\frac{\partial g_i(\kappa; p)}{\partial\left(\dfrac{1}{r+\kappa}\right)} > 0$$

if and only if

$$u_i'(a_i) < \sum_{j=1}^{I} \pi_{ij} u_j'(a_i). \tag{50}$$

(iii) *Consider the case* $I = 2$. *Then for* $i, j \in \{1, 2\}$ (*with* $j \neq i$),

$$\bar{u}_i(a) = \frac{r+\kappa+\delta\pi_{ji}}{r+\kappa+\delta(\pi_{12}+\pi_{21})}u_i(a) + \frac{\delta\pi_{ij}}{r+\kappa+\delta(\pi_{12}+\pi_{21})}u_j(a) \tag{51}$$

and $\partial g_i(\kappa; p)/\partial\kappa > 0$ *if and only if* $u_i'(a_i) > u_j'(a_i)$.

PROOF: (i) From (48), $\partial g_i(\kappa; p)/\partial\kappa$ has the sign of $\sum_{k=0}^{\infty}(\frac{\delta}{r+\kappa} - k)\mu_k\bar{u}_i^{(k)\prime}(a_i)$, so we sign the latter. Let $\bar{\mathbb{Z}} = \mathbb{Z} \cap (-\infty, \frac{\delta}{r+\kappa})$, where $\mathbb{Z}$ denotes the set of integers, and define $\bar{k} = \max_{k\in\bar{\mathbb{Z}}} k$. Suppose that (49) holds. Then $\{\bar{u}_i^{(k)\prime}\}_{k=0}^{\infty}$ is a decreasing sequence with $\bar{u}_i^{(0)\prime}(a_i) = u_i'(a_i) > \sum_{j=1}^{I} \pi_j^* u_j'(a_i) = \lim_{k\to\infty} \bar{u}_i^{(k)\prime}(a_i)$. Since $(\frac{\delta}{r+\kappa} - k)\mu_k > 0$ for $k < \bar{k}+1$ and $(\frac{\delta}{r+\kappa} - k)\mu_k \leq 0$ for $k \geq \bar{k}+1$, the fact that $\bar{u}_i^{(k+1)\prime} \leq \bar{u}_i^{(k)\prime}$ for all k implies

$$\sum_{k=0}^{\bar{k}}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(\bar{k})\prime}(a_i) + \sum_{k=\bar{k}+1}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(\bar{k}+1)\prime}(a_i) \leq \sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i).$$

Since $\sum_{k=0}^{\infty}(\frac{\delta}{r+\kappa}-k)\mu_k = 0$, the above inequality can be written as

$$0 \leq \left[\bar{u}_i^{(\bar{k})\prime}(a_i) - \bar{u}_i^{(\bar{k}+1)\prime}(a_i)\right]\sum_{k=0}^{\bar{k}}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \leq \sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i). \tag{52}$$

If $\bar{u}_i^{(\bar{k}+1)\prime}(a_i) < \bar{u}_i^{(\bar{k})\prime}(a_i)$, then the first inequality in (52) is strict. Alternatively, if $\bar{u}_i^{(\bar{k}+1)\prime}(a_i) = \bar{u}_i^{(\bar{k})\prime}(a_i)$, then the second inequality is strict, since $\bar{u}_i^{(0)\prime}(a_i) > \lim_{k\to\infty} \bar{u}_i^{(k)\prime}(a_i)$, which implies that

$$\sum_{k=0}^{\bar{k}}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(\bar{k})\prime}(a_i) < \sum_{k=0}^{\bar{k}}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i)$$

or

$$\sum_{k=\bar{k}+1}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(\bar{k}+1)\prime}(a_i) < \sum_{k=\bar{k}+1}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i)$$

must hold. In any case, $\partial g_i(\kappa; p)/\partial\kappa > 0$ follows. Conversely, suppose that

$$\sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i) > 0,$$

but (49) does not hold, that is, $u_i'(a_i) \leq \sum_{j=1}^{I}\pi_j^* u_j'(a_i)$. Then $\{\bar{u}_i^{(k)\prime}\}_{k=0}^{\infty}$ is an increasing sequence and

$$\sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i) \leq \sum_{k=0}^{\bar{k}}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(\bar{k})\prime}(a_i) + \sum_{k=\bar{k}+1}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(\bar{k}+1)\prime}(a_i).$$

This leads to

$$\sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \bar{u}_i^{(k)\prime}(a_i)$$
$$\leq \left[\bar{u}_i^{(\bar{k})\prime}(a_i)-\bar{u}_i^{(\bar{k}+1)\prime}(a_i)\right]\sum_{k=0}^{\bar{k}}\left(\frac{\delta}{r+\kappa}-k\right)\mu_k \leq 0,$$

a contradiction.

(ii) Let $\varkappa=(r+\kappa)^{-1}$ and differentiate (47) with respect to $\varkappa$ (with p given) to find

$$\frac{\partial g_i\left(\dfrac{1}{\varkappa}-r;p\right)}{\partial\varkappa} = \frac{\dfrac{1}{(1+\delta\varkappa)\varkappa}\displaystyle\sum_{k=0}^{\infty}(k-\delta\varkappa)\left(\frac{1}{1+\delta\varkappa}\right)\left(\frac{\delta\varkappa}{1+\delta\varkappa}\right)^k \bar{u}_i^{(k)\prime}(a_i)}{-\bar{u}_i''(a_i)}.$$

The numerator can be written as

$$\delta\left(\frac{1}{1+\delta\varkappa}\right)^2\left[\frac{1-\delta\varkappa}{1+\delta\varkappa}\bar{u}_i^{(1)\prime}(a_i)-\bar{u}_i^{(0)\prime}(a_i)+O(\varkappa)\right],$$

where $O(\varkappa)=\sum_{k=2}^{\infty}(k-\delta\varkappa)(\frac{1}{1+\delta\varkappa})^k(\delta\varkappa)^{k-1}\bar{u}_i^{(k)\prime}(a_i)$. Since $\lim_{\varkappa\to 0}O(\varkappa)=0$, we have

$$\lim_{\varkappa\to 0}\frac{\partial g_i\left(\dfrac{1}{\varkappa}-r;p\right)}{\partial\varkappa}=\frac{\delta[\bar{u}_i^{(1)\prime}(a_i)-\bar{u}_i^{(0)\prime}(a_i)]}{-\bar{u}_i''(a_i)}.$$

Finally, $\bar{u}_i^{(0)\prime}(a_i)=u_i'(a_i)$ and $\bar{u}_i^{(1)\prime}(a_i)=\sum_{j=1}^{I}\pi_{ij}u_j'(a_i)$ imply that

$$\lim_{\kappa\to\infty}\frac{\partial g_i(\kappa;p)}{\partial\left(\dfrac{1}{r+\kappa}\right)}>0$$

if and only if (50) holds.

(iii) Let $I=2$. For $i=1$, (40) reduces to

$$\bar{u}_1(a)=\left(\frac{r+\kappa}{r+\kappa+\delta}\right)\sum_{k=0}^{\infty}\left(\frac{\delta}{r+\kappa+\delta}\right)^k$$
$$\times\left[\pi_{11}^{(k)}u_1(a)+\left(1-\pi_{11}^{(k)}\right)u_2(a)\right],$$

where

$$\pi_{11}^{(k)} = \frac{\pi_{21}}{\pi_{12}+\pi_{21}} + \frac{\pi_{12}}{\pi_{12}+\pi_{21}}(1-\pi_{12}-\pi_{21})^k,$$

since $\pi_{12}+\pi_{21}>0$. Collect terms to arrive at (51) for $i=1$. The expression for $i=2$ is obtained similarly. The first-order condition (47) specializes to

$$\frac{r+\kappa+\delta\pi_{ji}}{r+\kappa+\delta(\pi_{12}+\pi_{21})}u_i'(a_i) + \frac{\delta\pi_{ij}}{r+\kappa+\delta(\pi_{12}+\pi_{21})}u_j'(a_i) = rp.$$

This can be differentiated with respect to κ (for fixed p) to obtain

$$\frac{\partial g_i(\kappa;p)}{\partial\kappa} = \frac{\delta\pi_{ij}[u_i'(a_i)-u_j'(a_i)]}{-\bar{u}_i''(a_i)[r+\kappa+\delta(\pi_{12}+\pi_{21})]^2}.$$

This concludes the proof. *Q.E.D.*

For the i.i.d. case analyzed in the body of the paper, we found that if trading frictions decrease, an investor increases his asset holdings if his current marginal valuation exceeds his expected marginal valuation over the expected holding period (condition (23)). Proposition 6 extends this result and shows that the key insight does not rely on the preference shocks being i.i.d. For the case of multiplicative preference shocks we analyzed in Section 4, for example, we have $\bar{u}_i(a)=\bar{\varepsilon}_i u(a)$, with

$$\bar{\varepsilon}_i = \sum_{k=0}^{\infty}\mu_k\bar{\varepsilon}_i^{(k)} \tag{53}$$

and $\bar{\varepsilon}_i^{(k)} = \sum_{j=1}^{I}\pi_{ij}^{(k)}\varepsilon_j$. Note that $\lim_{k\to\infty}\bar{\varepsilon}_i^{(k)} = \sum_{j=1}^{I}\pi_j^*\varepsilon_j \equiv \bar{\varepsilon}$. Part (i) of Proposition 6 establishes that if this convergence is monotonic for i, then an investor with preference type i increases his asset holdings if and only if $\varepsilon_i>\bar{\varepsilon}$. This is essentially the same condition we derived in the i.i.d. case where $\pi_{ij}^{(k)}=\pi_j^*$ for all i and all $k\geq 1$. For this multiplicative case, the condition in part (ii) of the proposition reduces to $\varepsilon_i>\bar{\varepsilon}_i^{(1)}$, and if we let $\delta_{ij}\equiv\delta\pi_{ij}$ for $i\neq j$ and $\delta_{ii}\equiv\delta(1-\pi_{ii})$, it can be written as

$$\varepsilon_i > \frac{\sum_{j\neq i}\delta_{ij}\varepsilon_j}{\sum_{j\neq i}\delta_{ij}}. \tag{54}$$

Proposition 6 parallels Proposition 2 in Gârleanu (2009). Notation aside, (54) is identical to the condition in part (i) of his Proposition 2. The monotonicity condition in part (ii) of his proposition plays the role of the monotonicity condition in part (i) of ours. The two-valuation case in part (iii) of his proposition parallels part (iii) in ours.

An implication of the i.i.d. case that does not generalize is that if $\varepsilon_i < \varepsilon_j$ and the agent with preference type i increases his asset holdings in response to an increase in κ, then so does the agent with preference type j.[23] The robust insight instead is that an investor whose current marginal valuation is large—in the sense that it exceeds his expected marginal valuation over the expected holding period—increases his asset holdings if κ increases.

The following proposition characterizes the equilibrium price for a particular class of utility functions and generalizes the discussion that followed Proposition 5. Just as in the i.i.d. case, this price is independent of frictions as summarized by κ if the individual asset demand is linear in the idiosyncratic valuation (as is the case with logarithmic preferences).

PROPOSITION 7: *Let $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$. Then*

$$p = \frac{\left(\sum_{i=1}^{I} \pi_i^* \bar{\varepsilon}_i^{1/\sigma}\right)^{\sigma}}{rA^{\sigma}},$$

where $\bar{\varepsilon}_i = \sum_{k=0}^{\infty} \sum_{j=1}^{I} \mu_k \pi_{ij}^{(k)} \varepsilon_j$. If $u_i(a) = \varepsilon_i \ln a$, then

$$p = \frac{\sum_{j=1}^{I} \pi_j^* \varepsilon_j}{rA}.$$

PROOF: Since $u_i(a) = \varepsilon_i u(a)$, we have $\bar{u}_i(a) = \bar{\varepsilon}_i u(a)$ with $\bar{\varepsilon}_i$ given by (53), so (47) becomes $\bar{\varepsilon}_i u'(a_i) = rp$. The parametric assumption implies $a_i = (\bar{\varepsilon}_i/(rp))^{1/\sigma}$ so the steady-state market-clearing condition, $\sum_{i=1}^{I} \pi_i^* a_i = A$, yields the first expression for p. For $\sigma = 1$, $p = (rA)^{-1} \sum_{i=1}^{I} \pi_i^* \bar{\varepsilon}_i$, where

$$\sum_{i=1}^{I} \pi_i^* \bar{\varepsilon}_i = \sum_{j=1}^{I} \sum_{k=0}^{\infty} \sum_{i=1}^{I} \pi_i^* \pi_{ij}^{(k)} \mu_k \varepsilon_j = \sum_{j=1}^{I} \pi_j^* \varepsilon_j.$$

Q.E.D.

As in the i.i.d. case, it is difficult to sign the general equilibrium effects of α and η on trade volume in general. We provide analytical results for three cases.

[23] For example, with a more general process for preference shocks it is possible to have a parametrization $\{\varepsilon_i, \pi_{ij}\}_{i,j=1}^{I}$ with $\sum_{k=1}^{I} \pi_{ik} \varepsilon_k < \varepsilon_i < \varepsilon_j < \sum_{k=1}^{I} \pi_{jk} \varepsilon_k$, which according to part (ii) of Proposition 6 implies that, near the frictionless limit, the high valuation investor (the one with preference type ε_j) will reduce his asset holdings and the low valuation investor will increase his asset holdings if κ increases.

The first has $I = 2$ and a general preference specification, the second considers a market close to the frictionless limit, and the third considers a market with severe trading frictions.

PROPOSITION 8: (i) *Let* $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ *with* $\sigma > 0$ *and assume that* $I = 2$. *Trade volume increases with* κ.

(ii) *Let* $u_i(a) = \varepsilon_i \ln a$ *and suppose that* $\bar{\varepsilon}_j > \bar{\varepsilon}_i$ *implies* $\varepsilon_j - \sum_{k=1}^{I} \pi_{jk}\varepsilon_k > \varepsilon_i - \sum_{k=1}^{I} \pi_{ik}\varepsilon_k$ *for all* $i, j \in \mathbb{X}^2$. *Trade volume decreases with* η *in the frictionless limit* (*as* $\kappa \to \infty$).

(iii) *Trade volume approaches zero as* $r + \kappa \to 0$.

PROOF: (i) With $I = 2$,

$$n_{12} = n_{21} = \frac{\delta\pi_{12}\pi_{21}}{[\alpha + \delta(\pi_{12} + \pi_{21})](\pi_{12} + \pi_{21})},$$

so trade volume is

$$\mathcal{V} = \frac{\alpha\delta\pi_{12}\pi_{21}}{[\alpha + \delta(\pi_{12} + \pi_{21})](\pi_{12} + \pi_{21})}(a_2 - a_1).$$

The preference specification together with (51) implies $a_i = (\bar{\varepsilon}_i/(rp))^{1/\sigma}$ for $i = 1, 2$, where

$$\bar{\varepsilon}_1 = \frac{r + \kappa + \delta\pi_{21}}{r + \kappa + \delta(\pi_{12} + \pi_{21})}\varepsilon_1 + \frac{\delta\pi_{12}}{r + \kappa + \delta(\pi_{12} + \pi_{21})}\varepsilon_2$$

and

$$\bar{\varepsilon}_2 = \frac{r + \kappa + \delta\pi_{12}}{r + \kappa + \delta(\pi_{12} + \pi_{21})}\varepsilon_2 + \frac{\delta\pi_{21}}{r + \kappa + \delta(\pi_{12} + \pi_{21})}\varepsilon_1.$$

Since $rp = (\pi_1^* \bar{\varepsilon}_1^{1/\sigma} + \pi_2^* \bar{\varepsilon}_2^{1/\sigma})^\sigma / A^\sigma$,

$$a_i = \frac{\bar{\varepsilon}_i^{1/\sigma}}{\pi_1^* \bar{\varepsilon}_1^{1/\sigma} + \pi_2^* \bar{\varepsilon}_2^{1/\sigma}} A.$$

Differentiate this expression with respect to κ to find that $\partial a_2/\partial\kappa$ has the sign of $(\varepsilon_2 - \varepsilon_1)$ and $\partial a_1/\partial\kappa$ has the opposite sign. Since $\varepsilon_1 < \varepsilon_2$, $da_1/d\kappa < 0 < da_2/d\kappa$. To find $\frac{d\mathcal{V}}{d\kappa}$, we consider two cases. (a) An increase in κ caused by a decrease in η (keeping α constant). For this case,

$$\frac{d\mathcal{V}}{d\kappa} = \frac{da_2}{d\kappa} - \frac{da_1}{d\kappa} > 0.$$

(b) An increase in κ caused by an increase in α, which implies

$$\frac{d\mathcal{V}}{d\kappa}=\left[\frac{\delta}{\alpha+\delta(\pi_{12}+\pi_{21})}\right]^2\pi_{12}\pi_{21}(a_2-a_1) + \frac{\alpha\delta\pi_{12}\pi_{21}}{[\alpha+\delta(\pi_{12}+\pi_{21})](\pi_{12}+\pi_{21})}\left(\frac{da_2}{d\kappa}-\frac{da_1}{d\kappa}\right)>0.$$

(ii) Let $\varkappa=(r+\kappa)^{-1}$. Under $u_i(a)=\varepsilon_i\ln a$, (47) implies $a_i=\bar{\varepsilon}_i/(rp)$, where $\bar{\varepsilon}_i=\sum_{k=0}^{\infty}\mu_k\bar{\varepsilon}_i^{(k)}$ with $\bar{\varepsilon}_i^{(k)}=\sum_{j=1}^{I}\pi_{ij}^{(k)}\varepsilon_j$ and $\mu_k=(\frac{1}{1+\delta\varkappa})(\frac{\delta\varkappa}{1+\delta\varkappa})^k$. Differentiate with respect to $\varkappa$ to find

$$\frac{da_i}{d\varkappa}=\frac{1}{(1+\delta\varkappa)\varkappa}\sum_{k=0}^{\infty}(k-\delta\varkappa)\mu_k\bar{\varepsilon}_i^{(k)}. \tag{55}$$

We know from Proposition 7 that under this preference specification the equilibrium price is independent of κ, so (55) captures the general equilibrium effect of κ on a_i. Let $\varkappa\to 0$ as in part (ii) of the proof of Proposition 6 to find

$$\lim_{\varkappa\to 0}\frac{da_i}{d\varkappa}=\frac{\delta[\bar{\varepsilon}_i^{(1)}-\bar{\varepsilon}_i^{(0)}]}{rp}.$$

Therefore,

$$\frac{d(a_j-a_i)}{d\varkappa}=\frac{\delta}{rp}\left\{\bar{\varepsilon}_j^{(1)}-\varepsilon_j-\left[\bar{\varepsilon}_i^{(1)}-\varepsilon_i\right]\right\}.$$

The assumption that $\varepsilon_j-\bar{\varepsilon}_j^{(1)}>\varepsilon_i-\bar{\varepsilon}_i^{(1)}$ if $\bar{\varepsilon}_j>\bar{\varepsilon}_i$ implies $d(a_j-a_i)/d\varkappa<0$ for $a_j>a_i$ and $d(a_j-a_i)/d\varkappa>0$ for $a_j<a_i$, so an increase in $\varkappa$ decreases the size of every trade. If the increase in $\varkappa$ is due to an increase in η (i.e., keeping α constant), then the weights n_{ij} in (24) remain constant and $\mathcal{V}$ decreases.

(iii) From (40), $\bar{u}_i(a)=\sum_{j=1}^{I}\omega_{ij}(\bar{r})u_j(a)$, where $\omega_{ij}(\bar{r})=\sum_{k=0}^{\infty}(\frac{\bar{r}}{\bar{r}+\delta})(\frac{\delta}{\bar{r}+\delta})^k\times\pi_{ij}^{(k)}$ and $\bar{r}=r+\kappa$. We first show that for any $\varepsilon>0$, $|\omega_{ij}(\bar{r})-\pi_j^*|<\varepsilon$ obtains for all $\bar{r}$ close enough to 0. For any $\bar{r}>0$ and any $N\in\mathbb{Z}_+$,

$$|\omega_{ij}(\bar{r})-\pi_j^*|\le\left|\sum_{k=0}^{N}\left(\frac{\bar{r}}{\bar{r}+\delta}\right)\left(\frac{\delta}{\bar{r}+\delta}\right)^k\left[\pi_{ij}^{(k)}-\pi_j^*\right]\right| + \left|\sum_{k=N+1}^{\infty}\left(\frac{\bar{r}}{\bar{r}+\delta}\right)\left(\frac{\delta}{\bar{r}+\delta}\right)^k\left[\pi_{ij}^{(k)}-\pi_j^*\right]\right|.$$

Since $\lim_{k\to\infty}\pi_{ij}^{(k)}=\pi_j^*$, choose N large enough so that the second term is strictly smaller than $\varepsilon/2$ for any $\bar{r}>0$. The first term is bounded above by

$|1-(\frac{\delta}{\bar{r}+\delta})^{N+1}|$, so it is strictly less than $\varepsilon/2$ for all $\bar{r}$ close enough to 0. Therefore, $\lim_{\bar{r}\to 0}\omega_{ij}(\bar{r})=\pi_j^*$ and $\lim_{\bar{r}\to 0}\bar{u}_i(a)=\sum_{j=1}^{I}\pi_j^* u_j(a)$ for every i. In turn, (39) approaches $\max_{a'\geq 0}[\sum_{j=1}^{I}\pi_j^* u_j(a')-q(t)a']$, so $a_i\to A$ for all i. With this, $\mathcal{V}\to 0$ as $r+\kappa\to 0$ is immediate from (24). *Q.E.D.*

Part (i) of Proposition 8 is a generalization of part (i) of Proposition 3. Part (ii) of Proposition 8 is analogous to part (ii) of Proposition 3. The focus of the former on the frictionless limit simplifies the analysis of the effects of trading frictions on individual asset demands (see, e.g., part (ii) of Proposition 6). The additional assumption is a condition on the speed with which preference shocks revert to their unconditional mean. For example, suppose $\bar{\varepsilon}_j>\bar{\varepsilon}_i$, which means that the expected marginal valuation over the holding period for an investor who currently has preference type j is larger than for an investor with preference type i. Then the assumption requires that the expected change in the marginal valuation after a single preference shock (e.g., $\varepsilon_j-\sum_{k=1}^{I}\pi_{jk}\varepsilon_k$ for the agent with preference type j) must be larger for the investor with the higher current expected valuation over the holding period. Part (iii) of Proposition 8 generalizes part (iii) of Proposition 3 as well as the notion—which for the i.i.d. case was proved in part (ii) of Proposition 2 and used in the proof of Proposition 4—that if the investor is patient, the influence of his current valuation at the time of the trade on his choice of asset holdings vanishes as the market becomes very illiquid. In other words, as $r+\kappa\to 0$, the distribution of asset holdings converges to a mass point at A and trade volume approaches zero. This has important implications for intermediation fees and dealer revenue: both approach zero as trade sizes vanish, just as in the i.i.d. case. Note that intermediation fees and revenue also go to zero as κ becomes large, so they are nonmonotonic functions of κ. Therefore, the nonmonotonicity results we established for i.i.d. preference shocks (Proposition 4) generalize. Finally, these nonmonotonicities can generate multiple steady-state equilibria, so the multiplicity that we find for the i.i.d. case (Proposition 8 in Lagos and Rocheteau (2008)) can also be generalized.

APPENDIX C: STRATEGIC BARGAINING

In the body of the paper we assumed that when an investor and a dealer trade, the new asset position of the investor, a', and the fee, ϕ, are the solution to a Nash bargaining problem where the dealer has bargaining power $\eta\in[0,1]$ and disagreement point $W(t)$, and the investor has disagreement point $V_i(a,t)$. In this appendix we describe a strategic bargaining game with a unique subgame perfect equilibrium outcome that coincides with the solution of the axiomatic Nash bargaining problem we have adopted.

Our theory is meant to model a fast-moving market where investors and dealers do not form long-lasting relationships, but rather contact each other at relatively high frequencies and must trade on the spot, instantaneously, before they part ways. With this in mind, consider the following natural and simple strategic bargaining game. Upon contact, with probability η, Nature selects the dealer to make an instantaneous take-it-or-leave-it offer, which the investor must either accept or reject on the spot. With complementary probability, Nature selects the investor to make an instantaneous take-it-or-leave-it offer, which the dealer must either accept or reject on the spot. The whole process is instantaneous, and the dealer and the investor part ways regardless of the outcome.[24]

Let $\langle a_i^1(t), \phi_i^1(a,t)\rangle$ denote the proposal that the dealer makes to an investor of type i who is holding a at time t and let $\langle a_i^2(t), \phi_i^2(a,t)\rangle$ denote the offer that the latter makes to the former. The set of offers that an investor of type i who is holding asset position a finds acceptable at time t is

$$\mathcal{A}_i^2(a,t) = \{(a',\phi) : V_i(a',t) - p(t)(a'-a) - \phi \geq V_i(a,t)\}.$$

Similarly, the set of offers that a dealer finds acceptable at time t is $\mathcal{A}^1 = \{(a',\phi) : \phi \geq 0\}$. If the dealer is selected as the proposer, he will offer

$$\langle a_i^1(t), \phi_i^1(a,t)\rangle = \arg\max_{(a',\phi)} \phi \mathbb{I}_{\mathcal{A}_i^2(a,t)}(a',\phi),$$

where the maximization is subject to $a' \geq 0$ and $\mathbb{I}_{\mathcal{A}_i^2(a,t)}(a',\phi)$ is an indicator function that is equal to 1 if $(a',\phi) \in \mathcal{A}_i^2(a,t)$. It is easy to see that $a_i^1(t) = a_i(t)$, where $a_i(t)$ is as in (3), and $\eta\phi_i^1(a,t) = \phi_i(a,t)$, where $\phi_i(a,t)$ is as in (4). If the investor makes the offer, he chooses

$$\begin{aligned}\langle a_i^2(t), \phi_i^2(a,t)\rangle = \arg\max_{(a',\phi)} \big\{ & [V_i(a',t) - p(t)(a'-a) - \phi]\mathbb{I}_{\mathcal{A}^1}(a',\phi) \\ & + [1 - \mathbb{I}_{\mathcal{A}^1}(a',\phi)]V_i(a,t)\big\},\end{aligned}$$

where the maximization is subject to $a' \geq 0$ and $\mathbb{I}_{\mathcal{A}^1}(a',\phi)$ is an indicator function that is equal to 1 if $(a',\phi) \in \mathcal{A}^1$. Hence, $a_i^2(t) = a_i(t)$ and $\phi_i^2(a,t) = 0$. Note that regardless of who gets selected to make the offer, the outcome of the negotiation is that the investor exits the meeting with asset position $a_i(t)$. The transaction fee equals $\phi_i(a,t)/\eta$ if the dealer makes the offer and equals 0 if the investor makes the offer, so the expected fee (before Nature decides who will make the offer) equals $\phi_i(a,t)$. It is easy to check that with these equilibrium outcomes, the investors' and dealers' value functions are just as in the body of the paper and all our results go through (subject to the obvious reinterpretation of $\phi_i(a,t)$ as an *expected* intermediation fee, which is inconsequential).

[24]This type of bargaining procedure has been used extensively in search models of money, for example, Burdett, Trejos, and Wright (2001), as well as in search models of the labor market, for example, Kiyotaki and Lagos (2007).

APPENDIX D: PRINCIPLE OF OPTIMALITY

Consider an investor who effectively contacts the market with Poisson intensity κ and who is subject to preference shocks with Poisson intensity δ. Let $\{T_n\}_{n=1}^{\infty}$ denote the sequence of contact times and let N_t denote the number of contacts over the time interval $[0, t)$. Similarly, let $\{T'_n\}_{n=1}^{\infty}$ denote the sequence of times at which he receives preference shocks. We adopt the convention that $T_0 = T'_0 = 0$. Define the function $k: \mathbb{R}_+ \to \mathbb{X}$, and interpret $k(t)$ as the investor's preference type at time t. The process for preference shocks implies $k(t) = k(T'_n)$ for $t \in [T'_n, T'_{n+1})$ for any integer $n \geq 0$. The realization $\omega = (N_t, k(t))_{t\in[0,\infty)}$ summarizes an investor's individual history of shocks. Let Ω be the set of all such histories. Similarly, let $\omega^t = (N_s, k(s))_{s\in[0,t]}$ denote a history of shocks up to time t and let Ω^t be the collection of all such histories. We work with the probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where $\mathcal{H}$ is an appropriate σ-field of subsets of Ω (e.g., the σ-field generated by Ω^t for all finite t), and $\mathbb{P}$ is the probability measure on $\mathcal{H}$ induced by the independent Poisson processes for preference shocks and effective contacts with the market. Let $\mathcal{H}^t \subseteq \mathcal{H}$ be a partition of Ω such that $H^t_{\varpi} \in \mathcal{H}^t$ is a set of histories that coincide over $[0, t]$, that is, $H^t_{\varpi} = \{\omega \in \Omega : \omega^t = \varpi \text{ for some } \varpi \in \Omega^t\}$. The σ-field generated by $\mathcal{H}^t$, denoted $\mathcal{F}^t$, captures the information available to the investor at time t, and the filtration $\{\mathcal{F}^t, t \in \mathbb{R}_+\}$ represents how information is revealed over time.

An *asset plan*, $\mathbf{a} = (\boldsymbol{a}_t)_{t\in[0,\infty)}$, for the investor is a set of functions $\boldsymbol{a}_t : \Omega \to [0, \bar{a}]$ for all $t \geq 0$, such that $\boldsymbol{a}_t$ is $\mathcal{F}^t$-measurable.[25] An asset plan $(\boldsymbol{a}_t)_{t\in[0,\infty)}$ is feasible if for every ω, $\boldsymbol{a}_0(\omega)$ equals the given initial asset holding of the investor and $\boldsymbol{a}_t(\omega) = \boldsymbol{a}_{T_n}(\omega)$ for all $t \in [T_n, T_{n+1})$. Let $\mathbb{A}$ denote the set of all feasible asset plans. Let $\mathcal{U}^M_{k(t)}(\cdot, t)$ be the utility functional over the time interval $[t, T_M]$ of an investor with preference type $k(t)$ at time t. His utility over the period $[t, T_M]$ from following asset plan $\mathbf{a} = (\boldsymbol{a}_s)_{s\in[0,\infty)}$ is

$$
\begin{aligned}
(56) \qquad & \mathcal{U}^M_{k(t)}(\mathbf{a}, t) \\
&= \mathbb{E}_t\Bigg[\int_t^{T_{N_t+1}} e^{-r(s-t)} u_{k(s)}(\boldsymbol{a}_t(\omega))\, ds \\
&\quad + \sum_{n=1}^{M-1} \int_{T_{N_t+n}}^{T_{N_t+n+1}} e^{-r(s-t)} u_{k(s)}\big(\boldsymbol{a}_{T_{N_t+n}}(\omega)\big)\, ds \\
&\quad - e^{-r(T_{N_t+1}-t)} p(T_{N_t+1})\big[\boldsymbol{a}_{T_{N_t+1}}(\omega) - \boldsymbol{a}_t(\omega)\big] \\
&\quad - \sum_{n=1}^{M-1} e^{-r(T_{N_t+n+1}-t)} p\big(T_{N_t+n+1}\big)\big[\boldsymbol{a}_{T_{N_t+n+1}}(\omega) - \boldsymbol{a}_{T_{N_t+n}}(\omega)\big]\Bigg],
\end{aligned}
$$

[25]The upper bound $\bar{a}$ is imposed for technical reasons (to ensure that the investor's utility is bounded above) and is chosen to be sufficiently large so that it does not affect the investor's decision.

where $\mathbb{E}_t$ is shorthand for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}^t]$.[26] The first M terms on the right side of (56) represent the expected discounted sum of utility flows from holding the asset position prescribed by the asset plan **a** over the time interval $[t, T_{N_t+M})$. The first term, for instance, is the expected utility from holding the asset position $\boldsymbol{a}_t(\omega)$ from the initial time t until the next time the investor gains effective access to the market, T_{N_t+1}. Similarly, each term in the summation represents the utility from holding the asset over the period $[T_{N_t+n}, T_{N_t+n+1})$, that is, between the effective contact number $N_t + n$ and the next one. The second M terms represent the expected net utility cost to the investor from readjusting his asset holdings at the times he contacts the market. The term on the second line of (56), for instance, is the (expected, discounted to time t) disutility the investor incurs to buy $\boldsymbol{a}_{T_{N_t+1}}(\omega)$ on his $(N_t + 1)$th effective contact with the market, net of the utility he gets from selling the assets he is holding at this time, $\boldsymbol{a}_t(\omega)$. In what follows, we will leave the dependence of the function $\boldsymbol{a}_t$ on ω implicit to simplify the notation. By the law of iterated expectations, the utility functional in (56) can be rewritten as

$$
\begin{aligned}
(57)\qquad & \mathcal{U}^M_{k(t)}(\mathbf{a}, t) \\
&= \frac{\bar{u}_{k(t)}(\boldsymbol{a}_t)}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right]\boldsymbol{a}_t - \mathbb{E}_t\left[e^{-r(T_{N_t+M}-t)} p\left(T_{N_t+M}\right)\boldsymbol{a}_{T_{N_t+M}}\right] \\
&\quad + \frac{1}{r+\kappa}\mathbb{E}_t\left\{\sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)}\left[\bar{u}_{k(T_{N_t+n})}\left(\boldsymbol{a}_{T_{N_t+n}}\right) - q\left(T_{N_t+n}\right)\boldsymbol{a}_{T_{N_t+n}}\right]\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
(58)\qquad & \bar{u}_{k(T_{N_t+n})}\left(\boldsymbol{a}_{T_{N_t+n}}\right) \equiv (r+\kappa)\mathbb{E}_{T_{N_t+n}} \int_{T_{N_t+n}}^{T_{N_t+n+1}} e^{-r(s-T_{N_t+n})} u_{k(s)}\left(\boldsymbol{a}_{T_{N_t+n}}\right) ds, \\
& q\left(T_{N_t+n}\right) \equiv (r+\kappa)\left[p\left(T_{N_t+n}\right) - \mathbb{E}_{T_{N_t+n}} e^{-r(T_{N_t+n+1}-T_{N_t+n})} p\left(T_{N_t+n+1}\right)\right].
\end{aligned}
$$

Notice that the function $\bar{u}_i(a)$ is as in (7), and since

$$
\mathbb{E}_{T_{N_t+n}} e^{-r(T_{N_t+n+1}-T_{N_t+n})} p\left(T_{N_t+n+1}\right) = \kappa \int_0^\infty e^{-(r+\kappa)s} p\left(T_{N_t+n}+s\right) ds,
$$

the function $q(t)$ is the one defined in (8). For any finite M and any t, the utility functional $\mathcal{U}^M_{k(t)}(\mathbf{a}, t)$ is well defined for any feasible asset plan **a**.[27]

[26]Notice that the stochastic process $\{T_n\}_{n=1}^\infty$ can be thought of as being a function of the process ω (since $(N_t)_{t\in[0,\infty)}$ is a right-continuous step function with jumps at $\{T_n\}_{n=1}^\infty$), so for any $\mathcal{F}^t$-measurable function $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, the expectation $\mathbb{E}[f(\omega)|\mathcal{F}^t]$ is also integrating over $\{T_n\}_{n=1}^\infty$.

[27]From (7), it is clear that the first term on the right side of (57) is a well behaved function of $\boldsymbol{a}_t$, which is itself a bounded and $\mathcal{F}^t$-measurable function. Since throughout the paper we have

Next, for any given nonnegative measurable price function $p(t)$, we define the infinite-horizon utility for the investor from following a feasible asset plan $\mathbf{a}$ by

$$\mathcal{U}_{k(t)}(\mathbf{a}, t) = \limsup_{M\to\infty} \mathcal{U}^M_{k(t)}(\mathbf{a}, t).$$

For any feasible asset plan, the sequence

$$\left\{\sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)}\bar{u}_{k(T_{N_t+n})}\big(\boldsymbol{a}_{T_{N_t+n}}\big)\right\}_{M=1}^{\infty}$$

has a limit. This limit may be a finite number or $-\infty$.[28] The sequence

$$\left\{\sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)}q\big(T_{N_t+n}\big)\boldsymbol{a}_{T_{N_t+n}}\right\}_{M=1}^{\infty}$$

is nondecreasing, so it has a limit, which may be $+\infty$. Let

$$f_M(\omega) \equiv \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)}\big[\bar{u}_{k(T_{N_t+n})}\big(\boldsymbol{a}_{T_{N_t+n}}\big) - q\big(T_{N_t+n}\big)\boldsymbol{a}_{T_{N_t+n}}\big]. \tag{59}$$

Then we have shown that $\lim_{M\to\infty} f_M$ exists (it may be finite or $-\infty$). If we rescale u_i for each i so that $u_i(\bar{a}) \le 0$ for all i, we see that the sequence $\{-f_M\}_{M=1}^{\infty}$ is a monotone increasing sequence of measurable functions that converge pointwise to $-\lim_{M\to\infty} f_M$, so by the monotone convergence theorem (e.g., Theorem 7.8 in Stokey and Lucas (1989)), we have $\lim_{M\to\infty}\mathbb{E}_t[f_M] = \mathbb{E}_t[\lim_{M\to\infty} f_M]$. All this implies that, given a price path $p(t)$, an investor's expected lifetime utility from following a feasible asset plan $\mathbf{a} = (\boldsymbol{a}_s)_{s\in[0,\infty)}$

specialized the analysis to price paths with the property that $p(t)$ is measurable, $q(t)$ is well defined for any t and the second term on the right side of (57) is well defined. Since $e^{-rt}p(t)\boldsymbol{a}_t$ is a nonnegative measurable function, the integral in the third term is well defined (although it need not be finite). As for the last term, notice that

$$\bar{u}_{k(T_{N_t+n})}(a) = \sum_{i=1}^{I}\bar{u}_i(a)\mathbb{I}_{\{k(T_{N_t+M})=i\}},$$

where $\bar{u}_i(a)$ is a continuous function for each i, so the integral of $e^{-rt}\bar{u}_{k(t)}(\boldsymbol{a}_t)$ is well defined. Finally, the integral of $q(t)\boldsymbol{a}_t$ is well defined since $p(t)$ and $\boldsymbol{a}_t$ are nonnegative and measurable.

[28]This limit is finite if u_i is bounded below for all i, since in that case we can rescale each utility function so that $u_i(0) \ge 0$ for all i, and the sequence of partial sums is nondecreasing and bounded above (because $a_t \le \bar{a}$ for all t and u_i is continuous for each i). Conversely, if some u_i is unbounded below, we can rescale u_i and every other u_j so that $u_k(\bar{a}) \le 0$ for all k. Then since the sequence of partial sums is nonincreasing, it has a limit, which could be $-\infty$.

is

$$
\begin{aligned}
&\mathcal{U}_{k(t)}(\mathbf{a}, t) \\
&\quad = \frac{\bar{u}_{k(t)}(\boldsymbol{a}_t)}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right]\boldsymbol{a}_t \\
&\qquad - \limsup_{M\to\infty} \mathbb{E}_t\left[e^{-r(T_{N_t+M}-t)} p\left(T_{N_t+M}\right)\boldsymbol{a}_{T_{N_t+M}}\right] \\
&\qquad + \frac{1}{r+\kappa}\mathbb{E}_t\left\{\sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)}\left[\bar{u}_{k(T_{N_t+n})}\left(\boldsymbol{a}_{T_{N_t+n}}\right) - q\left(T_{N_t+n}\right)\boldsymbol{a}_{T_{N_t+n}}\right]\right\},
\end{aligned}
$$

which is well defined for any feasible path.[29] The investor's problem at t is

$$
\max_{\mathbf{a}\in\mathbb{A}} \mathcal{U}_{k(t)}(\mathbf{a}, t), \quad \text{s.t.} \quad a_t = a \geq 0 \quad \text{and} \quad k(t) \in \mathbb{X} \text{ given.} \tag{60}
$$

The investor's maximum attainable utility is then

$$
V^*_{k(t)}(a, t) = \max_{\mathbf{a}\in\mathbb{A}} \mathcal{U}_{k(t)}(\mathbf{a}, t).
$$

PROPOSITION 9: *A feasible plan* $\mathbf{a}^* = (\boldsymbol{a}^*_s(\omega))_{s\in[t,\infty),\omega\in\Omega}$ *is optimal from a given initial date* $t \geq 0$ *if and only if it satisfies*

$$
\boldsymbol{a}^*_{T_n}(\omega) = \arg\max_{a\in[0,\bar{a}]}\left[\bar{u}_{k(T_n)}(a) - q(T_n)a\right] \quad \forall\omega \in \Omega, \forall\{T_n\}_{n=T_{N_t+1}}^{\infty} \tag{61}
$$

and

$$
\lim_{n\to\infty} \mathbb{E}_i\left\{e^{-r(T_{N_t+n}-t)} p\left(T_{N_t+n}\right)\boldsymbol{a}^*_{T_{N_t+n}}\right\} = 0. \tag{62}
$$

Moreover, if there exists a number $B > \max_j \bar{u}'_j(\infty)$ *such that* $q(s) \geq B$ *for all* s, *then an optimal plan exists and is unique.*

PROOF: The proof proceeds in three steps.

(i) We first show that (61) and (62) are sufficient for an optimum. Let $\mathbf{a}^*$ be the asset plan that satisfies (61) and (62), and let $\mathbf{a}$ be any other feasible plan.

[29] We have chosen to define the lifetime utility as $\limsup_{M\to\infty} \mathcal{U}^M(\mathbf{a}, t)$ rather than $\lim_{M\to\infty}\mathcal{U}^M(\mathbf{a}, t)$, because $\lim_{M\to\infty}\mathbb{E}_t[\exp(-r(T_{N_t+M} - t))p(T_{N_t+M})a_{T_{N_t+M}}]$ need not exist for every feasible asset plan. The definition we have adopted guarantees that the payoff from every feasible asset plan can be evaluated using the investor's utility function. As we show below, the optimal asset plan, $\mathbf{a}^*$, has the property that $\lim_{M\to\infty}\mathbb{E}_t[\exp(-r(T_{N_t+M} - t))p(T_{N_t+M})a^*_{T_{N_t+M}}] = 0$, which means that, equivalently, we could define the utility function as $\lim_{M\to\infty}\mathcal{U}^M(\mathbf{a}, t)$ and simply restrict the investor's choices to the set of feasible paths for which $\lim_{M\to\infty}\mathbb{E}_t[\exp(-r \times (T_{N_t+M} - t))p(T_{N_t+M})a_{T_{N_t+M}}]$ exists.

For any t, let $\Delta \equiv \mathcal{U}_{k(t)}(\mathbf{a}^*, t) - \mathcal{U}_{k(t)}(\mathbf{a}, t)$. Then

$$\begin{aligned}\Delta \geq{}& \frac{1}{r+\kappa}\mathbb{E}_t\left\{\sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)}\left[\bar{u}_{k(T_{N_t+n})}\left(\boldsymbol{a}^*_{T_{N_t+n}}\right) - q\left(T_{N_t+n}\right)\boldsymbol{a}^*_{T_{N_t+n}}\right]\right\} \\ &- \frac{1}{r+\kappa}\mathbb{E}_t\left\{\sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)}\left[\bar{u}_{k(T_{N_t+n})}\left(\boldsymbol{a}_{T_{N_t+n}}\right) - q\left(T_{N_t+n}\right)\boldsymbol{a}_{T_{N_t+n}}\right]\right\} \\ &- \limsup_{M\to\infty}\mathbb{E}_t\left[e^{-r(T_{N_t+M}-t)} p\left(T_{N_t+M}\right)\boldsymbol{a}^*_{T_{N_t+M}}\right].\end{aligned}$$

From (61) and (62), it follows that $\Delta \geq 0$.

(ii) Next we show that an optimal plan must satisfy (61) and (62). The first step is to notice that the objective function on the right side of (61) is strictly concave and differentiable, so $u_i'[\boldsymbol{a}_s^*(\omega)] - q(s) \leq 0$ ("=" if $\boldsymbol{a}_i^*(s) > 0$) is necessary and sufficient for an optimum. Since $q(s) > \bar{u}_i'(\infty)$ for all i, we can choose $\bar{a}$ large enough so that $q(s) > \bar{u}_i'(\bar{a})$ for all i and, therefore, (61) is the *unique* solution to the investor's problem at time s, for history ω, when his preference type is $k(s)$. Suppose that the asset plan $\tilde{\mathbf{a}}$ is optimal, with $\tilde{\boldsymbol{a}}_s(\omega) \neq \boldsymbol{a}_s^*(\omega)$ for some history ω at some date $s > t$. Since both $\tilde{\mathbf{a}}$ and $\mathbf{a}^*$ are feasible, $\tilde{\boldsymbol{a}}_{T_{N_s}}(\omega) \neq \boldsymbol{a}^*_{T_{N_s}}(\omega)$. Then the investor could maintain his asset plan $\tilde{\mathbf{a}}$ unchanged except at date T_{N_s} for history ω, where he could choose $\boldsymbol{a}^*_{T_{N_s}}(\omega)$. By (61), this deviation is feasible. Since the maximization in (61) has a unique solution, the proposed deviation strictly increases the investor's expected utility, so $\tilde{\mathbf{a}}$ could not have been optimal—a contradiction. Next, we show that any optimal policy must satisfy (62). Let $\mathbf{a}^*$ be an optimal plan and consider the feasible plan $(1-\varepsilon)\mathbf{a}^*$ for some small $\varepsilon > 0$. Let $\Delta_\varepsilon \equiv \mathcal{U}_{k(t)}(\mathbf{a}^*, t) - \mathcal{U}_{k(t)}[(1-\varepsilon)\mathbf{a}^*, t]$. Then

$$\begin{aligned}\Delta_\varepsilon ={}& \mathbb{E}_t\left\{\sum_{n=1}^{\infty} \frac{e^{-r(T_{N_t+n}-t)}}{r+\kappa}\left[\bar{u}_{k(T_{N_t+n})}\left(\boldsymbol{a}^*_{T_{N_t+n}}\right) - \bar{u}_{k(T_{N_t+n})}\left[(1-\varepsilon)\boldsymbol{a}^*_{T_{N_t+n}}\right]\right.\right. \\ &\left.\left. - \varepsilon q\left(T_{N_t+n}\right)\boldsymbol{a}^*_{T_{N_t+n}}\right]\right\} \\ &- \varepsilon \limsup_{M\to\infty}\mathbb{E}_t\left[e^{-r(T_{N_t+M}-t)} p\left(T_{N_t+M}\right)\boldsymbol{a}^*_{T_{N_t+M}}\right].\end{aligned}$$

Divide the previous expression by ε and take the limit as $\varepsilon \to 0$ (applying l'Hôpital's rule) to arrive at

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon}{\varepsilon} ={}& \frac{1}{r+\kappa}\mathbb{E}_t\left\{\sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)}\left[\bar{u}'_{k(T_{N_t+n})}\left(\boldsymbol{a}^*_{T_{N_t+n}}\right) - q\left(T_{N_t+n}\right)\right]\boldsymbol{a}^*_{T_{N_t+n}}\right\} \\ &- \limsup_{M\to\infty}\mathbb{E}_t\left[e^{-r(T_{N_t+M}-t)} p\left(T_{N_t+M}\right)\boldsymbol{a}^*_{T_{N_t+M}}\right].\end{aligned}$$

Since the asset plan $\mathbf{a}^*$ is optimal, the first-order condition for the investor's problem (61), that is, $[\bar{u}'_{k(T_n)}(\boldsymbol{a}^*_{T_n}) - q(T_n)]\boldsymbol{a}^*_{T_n} = 0$ for all $\{T_n\}_{n=T_{N_t+1}}^{\infty}$, implies

$$\lim_{\varepsilon\to 0} \frac{\Delta_\varepsilon}{\varepsilon} = -\limsup_{M\to\infty} \mathbb{E}_t\big[e^{-r(T_{N_t+M}-t)} p\big(T_{N_t+M}\big)\boldsymbol{a}^*_{T_{N_t+M}}\big]$$

and the optimality of $\mathbf{a}^*$ requires

$$0 \le -\limsup_{M\to\infty} \mathbb{E}_t\big[e^{-r(T_{N_t+M}-t)} p\big(T_{N_t+M}\big)\boldsymbol{a}^*_{T_{N_t+M}}\big].$$

Then, since $e^{-rT} p(T)\boldsymbol{a}^*_T \ge 0$ for all T, we have

$$\begin{aligned} 0 &\le \liminf_{M\to\infty} \mathbb{E}_t\big[e^{-r(T_{N_t+M}-t)} p\big(T_{N_t+M}\big)\boldsymbol{a}^*_{T_{N_t+M}}\big] \\ &\le \limsup_{M\to\infty} \mathbb{E}_t\big[e^{-r(T_{N_t+M}-t)} p\big(T_{N_t+M}\big)\boldsymbol{a}^*_{T_{N_t+M}}\big] \le 0, \end{aligned}$$

so the optimality of $\mathbf{a}^*$ requires

$$\lim_{M\to\infty} \mathbb{E}_t\big[e^{-r(T_{N_t+M}-t)} p\big(T_{N_t+M}\big)\boldsymbol{a}^*_{T_{N_t+M}}\big] = 0.$$

(iii) Finally, since the necessary conditions (61) and (62) determine a unique $\mathbf{a}^* = (\boldsymbol{a}^*_t(\omega))_{t\in[0,\infty),\omega\in\Omega}$, the optimal plan exists and is unique. *Q.E.D.*

The formulation we have laid out in this appendix is quite general in that it allows the investor to choose among feasible asset plans $\mathbf{a} = (\boldsymbol{a}_t(\omega))_{t\in[0,\infty),\omega\in\Omega}$, where $\boldsymbol{a}_t$ can be any $\mathcal{F}^t$-measurable function of the whole history of shocks, ω, as well as time, t. From (61), however, notice that the optimal asset plan $\mathbf{a}^* = (\boldsymbol{a}^*_t(\omega))_{t\in[0,\infty),\omega\in\Omega}$ is not history dependent: when the investor gains effective access to the market at time T_n, his optimal decision depends only on T_n and his preference type at that time, $k(T_n)$. For this reason, we can simplify the notation as we did in the body of the paper, by letting $a_{k(T_n)}(T_n) \equiv \boldsymbol{a}^*_{T_n}(\omega)$. With this notation, we can denote the optimal plan $\mathbf{a}^*$ simply by a sequence of functions $\{(a_i(t), t\in[0,\infty))\}_{i=1}^{I}$, with $a_i(t) = a_i(T_n)$ for all $t\in[T_n, T_{n+1})$ and every i. Also as in the body of the paper, we can use $\mathbb{E}_{k(t)}$ to denote $\mathbb{E}_t$, which stresses the fact that $k(t)$ summarizes all the relevant information available to the investor at time t that enables him to form the conditional expectation over ω. With this notation, consider an investor at time t, with asset holdings $a_t = a \ge 0$ and preference type $k(t) = i \in \mathbb{X}$ both given. His maximum attainable utility is $V_i^*(a,t) = \mathcal{U}_i(\mathbf{a}^*, t)$, that is,

$$(63)\qquad V_i^*(a,t) = \frac{\bar{u}_i(a)}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right]a + K_i(t),$$

where

$$K_i(t) = \mathbb{E}_i\left\{\sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)}\left[\frac{\bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})]}{r+\kappa} - \frac{q(T_{N_t+n})}{r+\kappa}a_{k(T_{N_t+n})}(T_{N_t+n})\right]\right\}.$$

From Proposition 9 we know that if there exists a number $B > \max_j \bar{u}'_j(\infty)$ such that $q(s) \geq B$ for all s, then an optimal plan $\{(a_i(t), t \in [0, \infty))\}_{i=1}^{I}$ exists and is unique, so $K_i(t)$ is well defined. If, in addition, there exists a real number $\overline{B}$ such that $q(t) \leq \overline{B}$ for all t, then $K_i(t) \in \mathbb{R}$ for all t and every i.

Instead of considering (60), in the body of the paper we described the investor's problem using a recursive functional equation (i.e., (1)) with asset holdings and fees given by (2), which we showed to be equivalent to (25). Lemma 8 formalizes the relationship between both formulations of the investor's problem, (25) and (60). Before we prove this result, it is convenient to establish a preliminary result.

LEMMA 7: *For any* $t \geq 0$,

$$K_{k(t)}(t) = \frac{1}{r+\kappa}\mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)}\big[\bar{u}_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}(T_{N_t+1})\big] - q(T_{N_t+1})a_{k(T_{N_t+1})}(T_{N_t+1})\big]\big\} + \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)}K_{k(T_{N_t+1})}(T_{N_t+1})\big].$$

PROOF: First, notice that for all integers $n \geq 0$, we have $N_s = N_t + n$ if $s = T_{N_t+n}$, so the definition of $K_{k(t)}(t)$ implies

$$K_{k(T_{N_t+1})}(T_{N_t+1}) = \frac{1}{r+\kappa}\mathbb{E}_{k(T_{N_t+1})}\left\{\sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})}\bar{u}_{k(T_{N_t+n})}\big[a_{k(T_{N_t+n})}(T_{N_t+n})\big]\right\} - \frac{1}{r+\kappa}\mathbb{E}_{k(T_{N_t+1})}\left\{\sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})}q(T_{N_t+n})a_{k(T_{N_t+n})}(T_{N_t+n})\right\}. \tag{64}$$

Also from the definition of $K_{k(t)}(t)$,

$$K_{k(t)}(t) = \frac{1}{r+\kappa}\mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)}\big[\bar{u}_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}(T_{N_t+1})\big] - q(T_{N_t+1})a_{k(T_{N_t+1})}(T_{N_t+1})\big]\big\}$$

$$
\begin{aligned}
&+\frac{1}{r+\kappa}\mathbb{E}_{k(t)}\left\{\sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-t)}\big[\bar{u}_{k(T_{N_t+n})}\big[a_{k(T_{N_t+n})}\big(T_{N_t+n}\big)\big]\right.\\
&\qquad\left. - q\big(T_{N_t+n}\big)a_{k(T_{N_t+n})}\big(T_{N_t+n}\big)\big]\right\}\\
&=\frac{1}{r+\kappa}\mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)}\big[\bar{u}_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\\
&\qquad - q\big(T_{N_t+1}\big)a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\big\}\\
&\quad+\mathbb{E}_{k(t)}e^{-r(T_{N_t+1}-t)}\left[\frac{1}{r+\kappa}\right.\\
&\quad\left.\times\mathbb{E}_{k(T_{N_t+1})}\left\{\sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})}\bar{u}_{k(T_{N_t+n})}\big[a_{k(T_{N_t+n})}\big(T_{N_t+n}\big)\big]\right\}\right]\\
&\quad-\mathbb{E}_{k(t)}e^{-r(T_{N_t+1}-t)}\left[\frac{1}{r+\kappa}\right.\\
&\quad\left.\times\mathbb{E}_{k(T_{N_t+1})}\left\{\sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})}q\big(T_{N_t+n}\big)a_{k(T_{N_t+n})}\big(T_{N_t+n}\big)\right\}\right]\\
&=\frac{1}{r+\kappa}\mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)}\big[\bar{u}_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\\
&\qquad - q\big(T_{N_t+1}\big)a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\big\}\\
&\quad+\mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)}K_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big].
\end{aligned}
$$

The last equality follows from (64). *Q.E.D.*

LEMMA 8: *Consider an investor who, at some initial time $t \geq 0$, starts with asset position a and preference type $k(t) \in \mathbb{X}$, and suppose that there exists a number $B > \max_j \bar{u}_j'(\infty)$ such that $q(s) \geq B$ for all $s \geq t$.*

(i) *The maximum value of* (60), *that is,* $V^*_{k(t)}(a, t)$, *satisfies the functional equation* (25).

(ii) *The asset plan that solves* (60), *that is,* $(a_{k(T_{N_s})}(s), s \in [t, \infty))$, *satisfies*

$$
\begin{aligned}
&V^*_{k(t)}\big[a_{k(T_{N_t})}\big(T_{N_t}\big), t\big]\\
&\quad=\frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r+\kappa}+\mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)}p\big(T_{N_t+1}\big)a_{k(T_{N_t})}\big(T_{N_t}\big)\big]\\
&\qquad+\mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)}\big\{V^*_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big), T_{N_t+1}\big]\\
&\qquad\qquad - p\big(T_{N_t+1}\big)a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big\}\big].
\end{aligned}
$$

(iii) *Let* $(a_{k(T_{N_s})}(s), s \in [t, \infty))$ *be the asset plan induced by* (25), *that is*, *the asset plan in* (6), *with*

$$\lim_{n\to\infty} \mathbb{E}_i\big[e^{-r(T_{N_t+n}-t)} p\big(T_{N_t+n}\big) a_{k(T_{N_t+n})}\big(T_{N_t+n}\big)\big] = 0 \tag{65}$$

for each $i \in \mathbb{X}$. *Then this asset plan achieves the maximum in* (60).

(iv) *Let* $(a_{k(T_{N_s})}(s), s \in [t, \infty))$ *be the asset plan induced by* (25) *and assume it satisfies* (65). *If* $V_i(a, t)$ *solves* (25) *and satisfies*

$$\lim_{n\to\infty} \mathbb{E}_i\big[e^{-r(T_{N_t+n}-t)} V_{k(T_{N_t+n})}\big[a_{k(T_{N_t+n})}\big(T_{N_t+n}\big), T_{N_t+n}\big]\big] = 0 \tag{66}$$

for each $i \in \mathbb{X}$, *then* $V_i(a, t) = V_i^*(a, t)$.

PROOF: (i) If we let $V^*(a, t) \equiv \{V_i^*(a, t)\}_{i=1}^I$ and regard the right side of (25) as a map F, we need to show $FV^* = V^*$. Substitute $V^*(a, t)$ as given by (63), into (25):

$$\begin{aligned}
&(FV^*)(a, t, i)\\
&= \frac{\bar{u}_i(a)}{r+\kappa} + \mathbb{E}_i\big[e^{-r(T_{N_t+1}-t)}\big\{p\big(T_{N_t+1}\big)a\\
&\qquad + \max_{a'\geq 0}\big[V^*_{k(T_{N_t+1})}\big(a', T_{N_t+1}\big) - p\big(T_{N_t+1}\big)a'\big]\big\}\big]\\
&= \frac{\bar{u}_i(a)}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right]a + \mathbb{E}_i\big[e^{-r(T_{N_t+1}-t)} K_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\\
&\quad + \frac{1}{r+\kappa}\mathbb{E}_i\big\{e^{-r(T_{N_t+1}-t)}\big[\bar{u}_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\\
&\qquad - q\big(T_{N_t+1}\big)a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\big\}\\
&= \frac{\bar{u}_i(a)}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right]a + K_i(t)\\
&= V_i^*(a, t),
\end{aligned}$$

where the third equality follows from Lemma 7.

(ii) From (63),

$$\begin{aligned}
&V^*_{k(t)}\big[a_{k(T_{N_t})}\big(T_{N_t}\big), t\big]\\
&= \frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right]a_{k(T_{N_t})}\big(T_{N_t}\big) + K_{k(t)}(t)\\
&= \frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r+\kappa} + \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)} p\big(T_{N_t+1}\big)a_{k(T_{N_t})}\big(T_{N_t}\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&+ K_{k(t)}(t)\\
&= \frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r+\kappa} + \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)} p\big(T_{N_t+1}\big) a_{k(T_{N_t})}\big(T_{N_t}\big)\big]\\
&\quad + \frac{1}{r+\kappa}\mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)}\big[\bar{u}_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\\
&\qquad\qquad - q\big(T_{N_t+1}\big) a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\big\}\\
&\quad + \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)} K_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\\
&= \frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r+\kappa} + \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)} p\big(T_{N_t+1}\big) a_{k(T_{N_t})}\big(T_{N_t}\big)\big]\\
&\quad + \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+1}-t)}\big\{V^*_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big), T_{N_t+1}\big]\\
&\qquad\qquad - p\big(T_{N_t+1}\big) a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big\}\big].
\end{aligned}
$$

The second equality follows from the definition of $q(t)$, the third equality follows from Lemma 7, and the fourth equality follows from the fact that

$$
\begin{aligned}
&V^*_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big), T_{N_t+1}\big]\\
&\quad = \frac{\bar{u}_{k(T_{N_t+1})}[a_{k(T_{N_t+1})}(T_{N_t+1})]}{r+\kappa}\\
&\qquad + \left[p(T_{N_t+1}) - \frac{q(T_{N_t+1})}{r+\kappa}\right] a_{k(T_{N_t+1})}\big(T_{N_t+1}\big) + K_{k(T_{N_t+1})}\big(T_{N_t+1}\big).
\end{aligned}
$$

(iii) Part (iii) is immediate from Proposition 9.

(iv) By (3) and (6), we can write (25) as

$$
\begin{aligned}
V_{k(t)}(a,t) = {}& \frac{\bar{u}_{k(t)}(a)}{r+\kappa} + \mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)} p\big(T_{N_t+1}\big)\big[a - a_{k(T_{N_t+1})}\big(T_{N_t+1}\big)\big]\big\}\\
& + \mathbb{E}_{k(t)}\big\{e^{-r(T_{N_t+1}-t)} V_{k(T_{N_t+1})}\big[a_{k(T_{N_t+1})}\big(T_{N_t+1}\big), T_{N_t+1}\big]\big\}.
\end{aligned}
$$

Iterate this expression forward $M-1$ times (using the law of iterated expectations and (58)) to arrive at

$$
\begin{aligned}
(67)\qquad & V^M_{k(t)}(a,t)\\
&= \frac{\bar{u}_{k(t)}(a)}{r+\kappa} + \left[p(t) - \frac{q(t)}{r+\kappa}\right] a\\
&\quad - \mathbb{E}_{k(t)}\big[e^{-r(T_{N_t+M}-t)} p\big(T_{N_t+M}\big) a_{k(T_{N_t+M})}\big(T_{N_t+M}\big)\big]
\end{aligned}
$$

$$+\frac{1}{r+\kappa}\mathbb{E}_{k(t)}\left\{\sum_{n=1}^{M-1}e^{-r(T_{N_t+n}-t)}\left[\bar{u}_{k(T_{N_t+n})}\left[a_{k(T_{N_t+n})}\left(T_{N_t+n}\right)\right]\right.\right.$$
$$\left.\left.-q\left(T_{N_t+n}\right)a_{k(T_{N_t+n})}\left(T_{N_t+n}\right)\right]\right\}$$
$$+\mathbb{E}_{k(t)}\left[e^{-r(T_{N_t+M}-t)}V_{k(T_{N_t+M})}\left[a_{k(T_{N_t+M})}\left(T_{N_t+M}\right),T_{N_t+M}\right]\right].$$

A function $V_{k(t)}(a,t)$ that solves (25) must satisfy (67) for all M, so the solution is $V_{k(t)}(a,t)=\lim_{M\to\infty}V^M_{k(t)}(a,t)$, provided this limit exists. From (67),

$$\lim_{M\to\infty}V^M_{k(t)}(a,t)$$
$$=\frac{\bar{u}_{k(t)}(a)}{r+\kappa}+\left[p(t)-\frac{q(t)}{r+\kappa}\right]a$$
$$-\lim_{M\to\infty}\mathbb{E}_{k(t)}\left[e^{-r(T_{N_t+M}-t)}p\left(T_{N_t+M}\right)a_{k(T_{N_t+M})}\left(T_{N_t+M}\right)\right]$$
$$+\lim_{M\to\infty}\mathbb{E}_{k(t)}\left\{\sum_{n=1}^{M-1}e^{-r(T_{N_t+n}-t)}\left[\frac{\bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})]}{r+\kappa}\right.\right.$$
$$\left.\left.-\frac{q(T_{N_t+n})}{r+\kappa}a_{k(T_{N_t+n})}\left(T_{N_t+n}\right)\right]\right\}$$
$$+\lim_{M\to\infty}\mathbb{E}_{k(t)}\left[e^{-r(T_{N_t+M}-t)}V_{k(T_{N_t+M})}\left[a_{k(T_{N_t+M})}\left(T_{N_t+M}\right),T_{N_t+M}\right]\right]$$
$$=\frac{\bar{u}_{k(t)}(a)}{r+\kappa}+\left[p(t)-\frac{q(t)}{r+\kappa}\right]a$$
$$+\lim_{M\to\infty}\mathbb{E}_{k(t)}\left\{\sum_{n=1}^{M-1}e^{-r(T_{N_t+n}-t)}\left[\frac{\bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})]}{r+\kappa}\right.\right.$$
$$\left.\left.-\frac{q(T_{N_t+n})}{r+\kappa}a_{k(T_{N_t+n})}\left(T_{N_t+n}\right)\right]\right\}.$$

The second equality follows from (65) and (66). Since $\{a_{k(T_{N_t+n})}(T_{N_t+n})\}_{n=1}^{\infty}$ is bounded and $\bar{u}_i$ is continuous for every i, $\{\bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})]\}_{n=1}^{\infty}$ is bounded above. Hence, without loss of generality, we can rescale u_i for each i so that $\{-\bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})]\}_{n=1}^{\infty}$ is a nonnegative sequence. Then the sequence $\{\bar{f}_M\}_{M=1}^{\infty}$, where

$$\bar{f}_M(\omega)\equiv\sum_{n=1}^{M-1}e^{-r(T_{N_t+n}-t)}\left[\bar{u}_{k(T_{N_t+n})}\left[a_{k(T_{N_t+n})}\left(T_{N_t+n}\right)\right]\right.$$
$$\left.-q\left(T_{N_t+n}\right)a_{k(T_{N_t+n})}\left(T_{N_t+n}\right)\right],$$

is a nonincreasing sequence and hence it has a limit, $\lim_{M\to\infty}\bar{f}_M$, which could be $-\infty$. Since $\{-\bar{f}_M\}_{M=1}^{\infty}$ is a monotone increasing sequence of measurable functions that converge pointwise to $-\lim_{M\to\infty}\bar{f}_M$, by the monotone convergence theorem (e.g., Theorem 7.8 in Stokey and Lucas (1989)), we have $\lim_{M\to\infty}\mathbb{E}_{k(t)}[\bar{f}_M]=\mathbb{E}_{k(t)}[\lim_{M\to\infty}\bar{f}_M]=(r+\kappa)K_{k(t)}(t)$ and, therefore, for every $k(t)\in\mathbb{X}$,

$$\lim_{M\to\infty}V_{k(t)}^{M}(a,t)=\frac{\bar{u}_{k(t)}(a)}{r+\kappa}+\left[p(t)-\frac{q(t)}{r+\kappa}\right]a+K_{k(t)}(t)=V_{k(t)}^{*}(a,t).$$

This concludes the proof. *Q.E.D.*

Lemma 8 establishes a version of Bellman's principle of optimality for the economy we analyze: Part (i) shows that $V_{k(t)}^{*}(a,t)$, the maximum value of the investor's problem given in (60), satisfies the functional equation (1) with asset holdings and fees given by (2) (which is equivalent to the functional equation (25)). Part (ii) establishes that the asset plan that solves (60) is an optimal plan implied by the functional equation (1) when this functional equation is evaluated at $V_{k(t)}^{*}(a,t)$. Part (iii) is a partial converse of part (ii): it proves that the asset plan that is optimal according to the functional equation (25) and that satisfies the boundedness condition (65) is the same asset plan that achieves the maximum of (60). Part (iv) is a partial converse of part (i): it shows that $V_{k(t)}^{*}(a,t)$ is the only solution of the functional equation (25) that satisfies the boundedness condition (66).

APPENDIX E: RELATED LITERATURE

In this appendix we draw connections to some related literature.

E.1. *Search Models of Over-the-Counter Markets*

Traders who operate in markets with OTC-style frictions will seek to mitigate these trading frictions by adjusting their asset positions so as to reduce their trading needs. Our analysis has shown that this is a critical aspect of investor behavior in illiquid markets. To illustrate this point, in this section we derive the main predictions of a version of DGP's model and contrast them with those of a special case of our formulation. This comparison will underscore the fact that the type of "liquidity hedging" that we have identified—and that only becomes possible with unrestricted asset holdings—generates new insights on how trading frictions shape the various dimensions of market liquidity, alters the empirical predictions of the theory, and leads to a different assessment of their normative implications.

We will contrast the empirical predictions of DGP's model with those of a special case of our model with $\mathbb{X}=\{1,2\}$ and $u_i(a)=\varepsilon_i a^{1-\sigma}/(1-\sigma)$ for $i\in\mathbb{X}$

and $\sigma > 0$. We focus on the version of DGP's model with no interinvestor meetings (e.g., the version that DGP use in their Theorem 4 and part (i) of Theorem 6). DGP restricted $a \in \{0, 1\}$, and let u_{ij} denote the flow utility of an investor with asset position $i \in \{0, 1\}$ and preference type $j \in \{0, 1\}$.[30] DGP assumed $u_{00} = u_{01} = 0$, so for comparison purposes, we do the same hereafter. To simplify the notation, in both models we let π denote the steady-state fraction of investors with high valuation.[31]

Price

Since asset holdings are indivisible in DGP, equilibrium in the interdealer market requires investors who are on the long side of the market to be indifferent between trading and not trading. It is easy to show that in steady state, investors who want to sell are on the short side if and only if $A < \pi$. The equilibrium price in the interdealer market is

$$p = \begin{cases} \dfrac{1}{r}\dfrac{(r+\kappa)u_{11} + \delta\bar{u}}{r+\kappa+\delta} & \text{if } A < \pi, \\ \dfrac{1}{r}\dfrac{(r+\kappa)u_{10} + \delta\bar{u}}{r+\kappa+\delta} & \text{if } \pi < A, \end{cases} \tag{68}$$

where $\bar{u} \equiv \pi_1 u_{11} + \pi_0 u_{10}$.[32]

The asset holding restrictions in DGP are also the reason why the asset price in their theory is independent of the stock of assets, A, for any $A < \pi$ and for any $A > \pi$, with a discontinuity at $A = \pi$. In contrast, the asset price in our model is smooth and decreasing in A. For example, in the special case of our model that we are considering in this section, $p = (\sum_i \pi_i \bar{\varepsilon}_i^{1/\sigma})^\sigma / r A^\sigma$.[33] The behavior of the asset price in response to changes in the trading frictions in DGP depends critically on the level of A. From (68), p is increasing in α (decreasing in η) if $A < \pi$, but decreasing in α (increasing in η) if $A > \pi$. In contrast, with unrestricted asset holdings these extensive-margin considerations are irrelevant to assess the impact of trading frictions on the asset price (recall Proposition 5).

[30] DGP stated their restriction on asset holdings as $a \in [0, 1]$ but only studied equilibria in which agents hold either 0 or 1 unit of the asset, which is effectively equivalent to imposing the restriction $a \in \{0, 1\}$.

[31] "High valuation" corresponds to the index 2 in our formulation and 1 in DGP.

[32] If $A = \pi$, $p \in [\frac{(r+\kappa)u_{10}+\delta\bar{u}}{r(r+\kappa+\delta)}, \frac{(r+\kappa)u_{11}+\delta\bar{u}}{r(r+\kappa+\delta)}]$ and the equilibrium price in the interdealer market is indeterminate.

[33] Notice that we obtain DGP's formulation with $A < \pi$ as a special case of ours when $\sigma \to 0$.

Trade Volume

Trade volume is

$$\mathcal{V}=\alpha\frac{\delta\pi(1-\pi)}{\alpha+\delta}\frac{(\bar{\varepsilon}_2)^{1/\sigma}-(\bar{\varepsilon}_1)^{1/\sigma}}{\pi(\bar{\varepsilon}_2)^{1/\sigma}+(1-\pi)(\bar{\varepsilon}_1)^{1/\sigma}}A$$

in our model and

$$\mathcal{V}_{\text{DGP}}=\alpha\frac{\delta\pi(1-\pi)}{\alpha+\delta}\min\left\{\frac{A}{\pi},\frac{1-A}{1-\pi}\right\}$$

in DGP. The latter is independent of the dealers' bargaining power, η, and of all preference parameters and holding payoffs. In contrast, these parameters are critical determinants of trade volume in our theory, as they influence the investors' choices of asset holdings (the second factor in $\mathcal{V}$). Our model predicts that markets in which dealers have less market power will tend to exhibit larger trade volume.[34]

Transaction Costs

DGP's transaction costs can be expressed in terms of the intermediation fees ϕ_{01} and ϕ_{10} that dealers charge investors who want to buy and sell, respectively. The equilibrium spread is $s=\eta(u_{11}-u_{10})/(r+\kappa+\delta)$.[35] Conditional on having contacted an investor, the expected intermediation fee that accrues to a dealer in DGP is $\Phi_{\text{DGP}}=\frac{\delta\pi(1-\pi)}{\alpha+\delta}\min\{\frac{A}{\pi},\frac{1-A}{1-\pi}\}s$. This key determinant of dealers' incentives to make markets is decreasing in the investors' contact rate with dealers, α, and increasing in the dealers' bargaining power, η. In contrast, as we have shown analytically in Proposition 4, in our model with no restrictions on asset holdings it is natural for the average fee to be nonmonotonic in α and η. Our theory suggests that this nonmonotonicity can be important. From an applied standpoint, it can help explain how OTC markets have reacted to recent changes in their market structure (see Lagos and Rocheteau (2006)).

[34] Apart from these qualitative differences, the theory with unrestricted portfolios also has different quantitative implications for the relationship between trade volume and trading frictions. For example, DGP's model has a sharp empirical implication: the elasticity of trade volume with respect to trading frictions equals $\frac{\delta}{\alpha+\delta}\in(0,1)$. In contrast, in the model with unrestricted asset holdings, the corresponding elasticity is larger by an amount that equals the elasticity of (a_2-a_1) with respect to α—which is positive, capturing the notion that each investor wishes to conduct a larger trade when frictions are reduced.

[35] Since asset holdings in DGP are restricted to lie in $\{0,1\}$, every trade is of size 1 and hence $\phi_{01}+\phi_{10}=s$. In addition, the indivisibility assumption implies that dealers either charge a fee on asset sales or on asset purchases, but not both. Specifically, if $A<\pi$, then $\phi_{01}=0$ and investors only pay a fee $\phi_{10}=s$ when they sell. Conversely, if $\pi<A$, $\phi_{10}=0$ and investors only pay a fee $\phi_{01}=s$ when they buy.

From a theoretical standpoint, it can be shown to generate self-fulfilling liquidity shortages in markets with free entry of dealers (see Proposition 8 in Lagos and Rocheteau (2008)).[36]

Another key difference with DGP is the fact that since the equilibrium in the model with unrestricted portfolios implies a nondegenerate distribution of trade sizes, our theory has predictions for the relationship between transaction costs and transaction sizes. As we showed in Lemma 4, transaction costs are increasing in the size of the transaction. Thus, if $a_i - a_j > a_i - a_k > 0$, then the effective price at which the investor buys is $\hat{p}_{ji} > \hat{p}_{ki}$, that is, he effectively pays higher prices when he conducts larger purchases. Conversely, $\hat{p}_{ji} < \hat{p}_{ki}$ if $a_i - a_j < a_i - a_k < 0$, that is, he effectively receives lower prices when he conducts larger sales. In other words, the theory with unrestricted asset holdings naturally generates instances of *price concession*, which are commonplace in OTC markets.[37]

Trading Delays

DGP endogenized trading delays by allowing a single monopolist dealer to choose search intensity once and for all at the beginning of time. Free entry of competing dealers or market-makers is a feature of most OTC markets; however, the implications of this microstructure have not yet been explored in the literature. We find that allowing for free entry of dealers is a natural way to endogenize trading delays and the amount of liquidity supplied by dealers, and that it provides an important channel through which changes in market conditions affect transaction costs and trade volume. In addition, the interaction between free entry and unrestricted asset holdings leads to a natural kind of strategic complementarity that can help rationalize self-fulfilling liquidity shortages in markets with OTC-style frictions (see Proposition 8 in Lagos and Rocheteau (2008)).

Welfare

The equilibrium allocation is always constrained to be efficient in the baseline model of DGP—regardless of the value of η—which stands in contrast to the finding we report in Proposition 2 in Lagos and Rocheteau (2008). The reason is that in our model investors choose asset holdings, while this intensive

[36]The spread, s, is decreasing in α and increasing in η in this version of DGP with no inter-investor meetings. One can also verify that the average effective spread weighted by the sizes of each trade and expressed as a proportion of the price is also decreasing in α and increasing in η. The behavior of this measure of the marketwide spread (i.e., (38) in Lagos and Rocheteau (2006)) is much more complicated in our model, where the investors' expected holding payoffs, their individual asset demands, the asset price, and the whole distribution of asset holdings change in response to a change in α. Our numerical work, some of which we have reported in Lagos and Rocheteau (2006), is in accordance with the predictions of DGP.

[37]See Section 4.3 in Harris (2003).

margin is absent in DGP. For the same reason, the inefficiency result we find in the context of the model with free entry also has no counterpart in DGP.

A paper that is closely related to ours is an independent contribution by Gârleanu (2008), which studied the asset pricing and volume implications of infrequent (Poisson) trading opportunities. Some of our findings are similar: he also finds that under certain conditions (e.g., a mean-reversion property of preference shocks), investors take more extreme positions when trading delays are short. Also, Gârleanu stressed that the asset price is not affected by the trading frictions—which is true in our model for a particular specification of the utility function (Proposition 5). In terms of differences, trades in Gârleanu (2008) are not intermediated by dealers, so he could not consider the implications of trading delays for transaction costs and dealers' incentives to provide liquidity, which are at the center of our analysis. Also, Gârleanu (2008) formalized the investors' motive for holding the asset by developing the "hedging needs" motive we mentioned in footnote 4. Despite the differences in the formulations, some of our results on the effects of α on trade volume are remarkably similar.[38]

E.2. *Search Models of Money*

Here we discuss the relationship between our theory and the search-theoretic literature on monetary exchange. In contrast to the monetary literature, our model does not have fiat money as an asset and it does not aim to explain the use or emergence of a medium of exchange. However, it shares a common objective with modern monetary theory, which is to endogenize some relevant dimensions of "liquidity." We organize the comparison around four types of results.

Endogenous Distribution of Asset Holdings

Because of idiosyncratic (trading) shocks, under incomplete markets, our model generates a nondegenerate distribution of wealth as in Green and Zhou (2002) and Molico (2006), but also Aiyagari (1994). The trading mechanism in our model is closer to the one in Molico: the asset is traded in bilateral matches and the transaction price is determined through bargaining. In terms of the methodology, both Aiyagari (1994) and Molico (2006) solved their models numerically. The model of Green and Zhou (2002) is closer to our analysis in that they can characterize the equilibrium and its distribution of money holdings analytically. Moreover, like us, they do not restrict their analysis to stationary equilibria. The pricing mechanism is different (Green and Zhou considered a double auction).

[38]See the discussion around Proposition 6 in online Appendix B for details.

Bargaining and the Distribution of Prices

A key insight of our model is that the intermediation fee depends on the (endogenous) asset position of the investor. Similarly, in monetary search models with bargaining, the transaction price depends on the traders' money balances. This dependence occurs through (at least) two channels. First, the buyer can be constrained by his money balances. This mechanism is present even in models with a degenerate distribution of money balances, such as Shi (1997) and Lagos and Wright (2005). Second, the money holdings of an agent affect his marginal utility of wealth and, hence, the terms of trade. These two effects are absent from our model, since our investors never face binding borrowing constraints and the marginal utility of wealth is normalized to one due to the quasi-linear preferences. An investor's asset holdings influence the outcome of the bargaining in our model because this asset position determines the size of the gains from trade that will be generated for readjusting the investor's asset holdings.

Uniqueness of the Equilibrium

The equilibrium (not just the steady state) is unique in our model. In contrast, the model of fiat money of Green and Zhou can display multiple equilibria. This indeterminacy is a general feature of models of fiat money. Even in models with a degenerate distribution of money balances (e.g., Lagos and Wright (2005)), the equilibrium is typically not unique, unless one restricts attention to steady-state monetary equilibria. Models of monetary exchange consider environments where the asset being traded is fiat money, whose value emerges endogenously when it is valued as a medium of exchange that mitigates a double coincidence of wants problem. In contrast, in our model and the rest of the literature that deals with the trading process in OTC markets, the asset being traded is not used to facilitate trades; it is valued for its intrinsic characteristics (e.g., dividend flow).

Endogenous Trading Delays and Multiple Equilibria

In our model, the multiplicity of steady-state equilibria with dealer entry arises from complementarities between investors' asset demands and dealers' entry decisions. If more dealers participate in the market, it is easier for investors to readjust their asset holdings, which induces them to take more extreme positions, and this in turn makes it profitable for dealers to enter. Rocheteau and Wright (2005) considered a monetary search model with free entry of sellers and found that the strategic complementarities between the sellers' entry decision and the buyers' demand for real balances generate multiple steady-state equilibria. If buyers accumulate more real balances, the buyer and the seller are able to exploit larger gains from trade, which gives more incentives for sellers to participate in the market. In both models, the multiplicity does not require increasing returns to scale in the matching function as in Diamond (1982) or as in most recent search models of financial markets (e.g., Vayanos and Weill (2008)). A key difference between our model and Rocheteau and Wright (2005) is the opportunity cost from holding real balances in the latter, which has no counterpart in our formulation. If the opportunity cost from holding cash balances to make a purchase is zero (e.g., if the nominal interest rate is zero), then the multiplicity of (active) steady-state equilibria in that model disappears. In contrast, the multiplicity in our model obtains even though investors do not bear any opportunity cost (e.g., forgone interest) while searching for an asset to purchase (since they have access to a technology to produce the numéraire good). Also notice that the gains from trade in Rocheteau and Wright (2005) depend on the mean of the distribution of real balances (since the distribution of real balances is degenerate as in Lagos and Wright (2005)), which is independent of trading frictions when the nominal interest rate is zero. In our model it is the second moment, which is endogenous and depends on the trading frictions, that gives rise to multiple steady-state equilibria.

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