



Some results on the optimality and implementation of the Friedman rule in the Search Theory of Money

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Abstract

I characterize a large family of monetary policies that implement Milton Friedman's prescription of zero nominal interest rates in a monetary search economy with multiple assets and aggregate uncertainty. This family of optimal policies is defined by two properties: (i) the money supply must be arbitrarily close to zero for an infinite number of dates, and (ii) asymptotically, on average (over the dates when fiat money plays an essential role), the growth rate of the money supply must be at least as large as the rate of time preference.

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Our final rule for the optimum quantity of money is that it will be attained by a rate of price deflation that makes the nominal rate of interest equal to zero.

Milton Friedman, "The optimum quantity of money" (1969) [3].

1. Introduction

Milton Friedman's prescription that monetary policy should induce a zero nominal interest rate in order to lead to an optimal allocation of resources, has come to be known as the *Friedman*

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rule. The cost of producing real balances is zero to the government, so the optimum quantity of real balances should be such that the marginal benefit is zero to the economic agents. Friedman's insight is so basic, so fundamental, that one would hope for his prescription to be valid regardless of the particular stance one assumes about the role that money plays in the economy. For some time now, the Friedman rule has been known to be optimal in competitive reduced-form monetary models under fairly broad conditions, whether money is introduced as an argument in the agents' utility functions, or through a cash-in-advance constraint. Recent developments in the Search Theory of Money have found that (absent extraneous, nonmonetary distortions) Friedman's rule is also optimal in environments where money is valued as a medium of exchange, and the mechanism of exchange is modeled explicitly.

One simple monetary policy that typically implements the Friedman rule is to contract the money supply at the rate of time preference. To date, this particular implementation of the Friedman rule has been the only one explored in search models of money—to the point that one may be led to the conclusion that contracting the money supply at the rate of time preference in fact *is* the Friedman rule in this class of models.

In this paper I characterize a large family of monetary policies that are necessary and sufficient to implement zero nominal interest rates (in the sense that these policies are consistent with the existence of a monetary equilibrium with zero nominal interest rates) in a version of Lagos and Wright [9], augmented to allow for aggregate liquidity shocks and a real financial asset that can be used as means of payment the same way money can. This family of optimal policies is defined by two properties: (i) the money supply must be arbitrarily close to zero for an infinite number of dates, and (ii) asymptotically, on average over the dates when fiat money plays an essential role, the growth rate of the money supply must be at least as large as the rate of time preference. For parametrizations such that the random value of the real asset is too low in every state of the world to satisfy the agents' liquidity needs, the family of optimal policies that I identify specializes to the class of monetary policies that Wilson [12] and Cole and Kocherlakota [2] have shown to be necessary and sufficient to implement the Friedman rule in the context of deterministic cash-in-advance economies. Given what was already known about the optimality of the Friedman rule in competitive reduced-form models, recent work in the Search Theory of Money has underscored the robustness of Friedman's basic insight and the ensuing prescription of zero nominal interest rates. The findings I report here, underscore the robustness of the characterization of a large class of monetary policies that implement Friedman's prescription.

2. The model

The model builds on [9,10]. Time is discrete, and the horizon infinite. There is a $[0, 1]$ continuum of infinitely lived agents. Each time period is divided into two subperiods where different activities take place. There are three nonstorable and perfectly divisible consumption goods at each date: *fruit*, *general goods*, and *special goods*. ("Nonstorable" means that the goods cannot be carried from one subperiod to the next.) Fruit and general goods are homogeneous goods, while special goods come in many varieties. The only durable commodity in the economy is a set of "Lucas trees." The number of trees is fixed and equal to the number of agents. Trees yield (the same amount of) a random quantity d_t of fruit in the second subperiod of every period t . The realization of the fruit dividend d_t becomes known to all at the beginning of period t (when agents enter the first subperiod). Production of fruit is entirely exogenous: no resources are utilized and it is not possible to affect the output at any time. The motion of d_t is described by a sequence of functions $F_t(s_{t+1}, s^t) = \Pr(d_{t+1} \leq s_{t+1} | d^t = s^t)$, where d^t denotes a history of

realizations of fruit dividends through period t , i.e., $d^t = (d_t, d_{t-1}, \dots, d_0)$. For each fixed s^t , $F_t(\cdot, s^t)$ is a distribution function with support $\mathcal{E} \subseteq (0, \infty)$.²

In each subperiod, every agent is endowed with \bar{n} units of time which can be employed as labor services. In the second subperiod, each agent has access to a linear production technology that transforms labor services into general goods. In the first subperiod, each agent has access to a linear production technology that transforms his own labor input into a particular variety of the special good that he himself does not consume. This specialization is modeled as follows. Given two agents i and j drawn at random, there are three possible events. The probability that i consumes the variety of special good that j produces but not vice-versa (a single coincidence) is denoted α . Symmetrically, the probability that j consumes the special good that i produces but not vice-versa is also α . In a single-coincidence meeting, the agent who wishes to consume is the *buyer*, and the agent who produces, the *seller*. The probability neither wants the good that the other can produce is $1 - 2\alpha$, with $\alpha \leq 1/2$. In contrast to special goods, fruit and general goods are homogeneous, and hence consumed (and in the case of general goods, also produced) by all agents.

In the first subperiod, agents participate in a *decentralized market* where trade is bilateral (each meeting is a random draw from the set of pairwise meetings), and the terms of trade are determined by bargaining. The specialization of agents over consumption and production of the special good combined with bilateral trade, give rise to a double-coincidence-of-wants problem in the first subperiod. In the second subperiod, agents trade in a *centralized market*. Agents cannot make binding commitments, and trading histories are private in a way that precludes any borrowing and lending between people, so all trade—both in the centralized and decentralized markets—must be *quid pro quo*.

Each tree has outstanding one durable and perfectly divisible equity share that represents the bearer's ownership, and confers him the right to collect the fruit dividends. There is a second financial asset, money, which is intrinsically useless (it is not an argument of any utility or production function), and unlike equity, ownership of money does not constitute a right to collect any resources. Money is issued by a "government" that at $t = 0$ commits to a *monetary policy*, represented by a sequence of positive real-valued functions, $\{\mu_t\}_{t=0}^\infty$. Given an initial stock of money, $M_0 > 0$, a monetary policy induces a *money supply process*, $\{M_t\}_{t=0}^\infty$, via $M_{t+1} = \mu_t(d^t)M_t$.³ The government injects or withdraws money via lump-sum transfers or taxes in the second subperiod of every period, i.e., along every sample path, $M_{t+1} = M_t + T_t$, where T_t is the lump-sum transfer (or tax, if negative). All assets are perfectly recognizable, cannot be forged, and can be

² This formulation encompasses several cases analyzed in the related literature. For example, if $F_t(s_{t+1}, s^t) = F(s_{t+1}, s_t)$ for all t , then d_t follows a time-homogeneous Markov process as in [10]. As another example, one could assume $\mathcal{E} = \{d_1, \dots, d_N\}$, and for each $s \in \mathcal{E}$, let $\pi_t(s|s^t) = \Pr(d_{t+1} = s|d^t = s^t)$ denote the probability of observing the realization s , conditional on the history of realizations s^t (with $\sum_{s \in \mathcal{E}} \pi_t(s|s^t) = 1$ for each s^t). This example is a special case of the above formulation, with $F_t(s_{t+1}, s^t) = \sum_{d_{t+1} \in \mathcal{E}(s_{t+1})} \pi_t(d_{t+1}|s^t)$, where $\mathcal{E}(s_{t+1}) = \{d \in \mathcal{E} : d \leq s_{t+1}\}$.

³ In general, a monetary policy induces a stochastic process $\{M_t\}_{t=0}^\infty$, i.e., a collection of random variables, M_t , defined on an appropriate probability space (see Appendix A for more details). As a special case, a deterministic monetary policy, that is, the case where $\{\mu_t\}_{t=0}^\infty$ is a sequence of positive constants, induces a *deterministic* money supply process, i.e., a deterministic sequence, $\{M_t\}_{t=0}^\infty$. Propositions 1 and 2 are proven for a general stochastic money supply process, while the characterization in Proposition 3 focuses on the case where $\{M_t\}_{t=0}^\infty$ is a deterministic money supply process. See [6] for a characterization of a large class of stochastic money supply processes that implement the Friedman rule in a stationary version of this environment.

traded among agents both in the centralized and decentralized markets.⁴ At $t = 0$ each agent is endowed with a_0^s equity shares and a_0^m units of fiat money.

Let the utility function for special goods, $u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and the utility function for fruit, $U : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, be continuously differentiable, increasing, and strictly concave, with $u(0) = U(0) = 0$, and let U be bounded. Let $-n$ be the utility from working n hours in the first subperiod. Also, suppose there exists $q^* \in (0, \infty)$ defined by $u'(q^*) = 1$, with $q^* \leq \bar{n}$. Let both, the utility for general goods, and the disutility from working in the second subperiod, be linear. The agents rank consumption and labor sequences according to

$$\liminf_{T \rightarrow \infty} E_0 \left\{ \sum_{t=0}^T \beta^t [u(q_t) - n_t + U(c_t) + y_t - h_t] \right\}, \tag{1}$$

where $\beta \in (0, 1)$, q_t and n_t are the quantities of special goods consumed and produced in the decentralized market, c_t denotes consumption of fruit, y_t consumption of general goods, h_t the hours worked in the second subperiod, and E_t is an expectations operator conditional on the information available to the agent at time t , defined with respect to the matching probabilities and the probability measure over sequences of dividends and money supplies induced by the sequence of transition functions, $\{F_t\}_{t=0}^\infty$, and the monetary policy, $\{\mu_t\}_{t=0}^\infty$.⁵

3. Equilibrium

Let $\mathbf{a}_t = (a_t^s, a_t^m)$ denote the portfolio of an agent who holds a_t^s shares and a_t^m dollars. Let $W_t(\mathbf{a}_t)$ and $V_t(\mathbf{a}_t)$ be the maximum attainable expected discounted utility of an agent when he enters the centralized, and decentralized market, respectively, at time t with portfolio \mathbf{a}_t . Then,

$$\begin{aligned} W_t(\mathbf{a}_t) &= \max_{c_t, y_t, h_t, \mathbf{a}_{t+1}} \left\{ U(c_t) + y_t - h_t + \beta E_t V_{t+1}(\mathbf{a}_{t+1}) \right\}, \\ \text{s.t. } c_t + w_t y_t + \phi_t^s a_{t+1}^s + \phi_t^m a_{t+1}^m &= (\phi_t^s + d_t) a_t^s + \phi_t^m (a_t^m + T_t) + w_t h_t, \\ 0 \leq c_t, \quad 0 \leq h_t \leq \bar{n}, \quad 0 \leq \mathbf{a}_{t+1}. \end{aligned} \tag{2}$$

The agent chooses consumption of fruit (c_t), consumption of general goods (y_t), labor supply (h_t), and an end-of-period portfolio (\mathbf{a}_{t+1}). Fruit is used as numéraire: w_t is the relative price of the general good, ϕ_t^s is the (ex-dividend) price of a share, and $1/\phi_t^m$ the dollar price of fruit. Substitute the budget constraint into the objective and rearrange to arrive at:

$$W_t(\mathbf{a}_t) = \lambda_t \mathbf{a}_t + \tau_t + \max_{c_t \geq 0} \left[U(c_t) - \frac{c_t}{w_t} \right] + \max_{\mathbf{a}_{t+1} \geq 0} \left[-\frac{\phi_t \mathbf{a}_{t+1}}{w_t} + \beta E_t V_{t+1}(\mathbf{a}_{t+1}) \right], \tag{4}$$

⁴ Lagos and Rocheteau [7] was the first paper to extend [9] to allow for another asset that competes with money as a medium of exchange. In [5] I consider a real version of [9] with aggregate uncertainty, in which equity shares and government bonds can serve as means of payment, and quantify the extent to which a liquidity premium can help to explain the equity premium and the risk-free rate puzzles. While the formulation I am studying in this paper has only two financial assets, equity and money, it would not be difficult to extend the results to an environment with a richer asset structure.

⁵ I follow Brock [1] and use this “overtaking criterion” to rank sequences of consumption and labor because $\lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t [u(q_t) - n_t + U(c_t) + y_t - h_t]$ may not be well defined for some feasible sequences. The same criterion was adopted by Wilson [12] and Cole and Kocherlakota [2] in their studies of competitive monetary economies subject to cash-in-advance constraints, and by Green and Zhou [4] in their study of dynamic monetary equilibria in a random matching economy.

where $\tau_t = \lambda_t^m T_t$, $\phi_t = (\phi_t^s, \phi_t^m)$, and $\lambda_t = (\lambda_t^s, \lambda_t^m)$, with

$$\lambda_t^s \equiv \frac{1}{w_t}(\phi_t^s + d_t) \quad \text{and} \quad \lambda_t^m \equiv \frac{1}{w_t}\phi_t^m. \tag{5}$$

Let $[q_t(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{p}_t(\mathbf{a}, \tilde{\mathbf{a}})]$ denote the terms at which a buyer who owns portfolio \mathbf{a} trades with a seller who owns portfolio $\tilde{\mathbf{a}}$, where $q_t(\mathbf{a}, \tilde{\mathbf{a}}) \in \mathbb{R}^+$ is the quantity of special good traded, and $\mathbf{p}_t(\mathbf{a}, \tilde{\mathbf{a}}) = [p_t^s(\mathbf{a}, \tilde{\mathbf{a}}), p_t^m(\mathbf{a}, \tilde{\mathbf{a}})] \in \mathbb{R}^+ \times \mathbb{R}^+$ is the transfer of assets from the buyer to the seller (the first argument is the transfer of equity). Consider a meeting in the decentralized market of period t , between a buyer with portfolio \mathbf{a}_t and a seller with portfolio $\tilde{\mathbf{a}}_t$. The terms of trade, (q_t, \mathbf{p}_t) , are determined by Nash bargaining where the buyer has all the bargaining power:

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} [u(q_t) + W_t(\mathbf{a}_t - \mathbf{p}_t) - W_t(\mathbf{a}_t)] \quad \text{s.t.} \quad W_t(\tilde{\mathbf{a}}_t + \mathbf{p}_t) - q_t \geq W_t(\tilde{\mathbf{a}}_t).$$

The constraint $\mathbf{p}_t \leq \mathbf{a}_t$ indicates that the buyer in a bilateral meeting cannot spend more assets than he owns. Since $W_t(\mathbf{a}_t + \mathbf{p}_t) - W_t(\mathbf{a}_t) = \lambda_t \mathbf{p}_t$, the bargaining problem is

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} [u(q_t) - \lambda_t \mathbf{p}_t] \quad \text{s.t.} \quad \lambda_t \mathbf{p}_t - q_t \geq 0.$$

If $\lambda_t \mathbf{a}_t \geq q^*$, the buyer buys $q_t = q^*$ in exchange for a vector \mathbf{p}_t of assets with real value $\lambda_t \mathbf{p}_t = q^* \leq \lambda_t \mathbf{a}_t$. Else, he pays the seller $\mathbf{p}_t = \mathbf{a}_t$, in exchange for $q_t = \lambda_t \mathbf{a}_t$. Hence, the quantity of output exchanged is $q_t(\mathbf{a}_t, \tilde{\mathbf{a}}_t) = \min(\lambda_t \mathbf{a}_t, q^*) \equiv q(\lambda_t \mathbf{a}_t)$, and the real value of the portfolio used as payment is $\lambda_t \mathbf{p}_t(\mathbf{a}_t, \tilde{\mathbf{a}}_t) = q(\lambda_t \mathbf{a}_t)$.

With the bargaining solution and the fact that $W_t(\mathbf{a}_t)$ is affine, the value of search to an agent who enters the decentralized market of period t with portfolio \mathbf{a}_t can be written as

$$V_t(\mathbf{a}_t) = S(\lambda_t \mathbf{a}_t) + W_t(\mathbf{a}_t), \tag{6}$$

where $S(x) \equiv \alpha\{u[q(x)] - q(x)\}$ is the expected gain from trading in the decentralized market.⁶ Substitute (6) into (4) to arrive at

$$W_t(\mathbf{a}_t) = \lambda_t \mathbf{a}_t + \tau_t + \max_{c_t \geq 0} \left[U(c_t) - \frac{c_t}{w_t} \right] + \max_{\mathbf{a}_{t+1} \geq 0} \left\{ -\frac{\phi_t \mathbf{a}_{t+1}}{w_t} + \beta E_t [S(\lambda_{t+1} \mathbf{a}_{t+1}) + W_{t+1}(\mathbf{a}_{t+1})] \right\}. \tag{7}$$

The agent’s problem consists of choosing a *feasible plan* $\{c_t, \mathbf{x}_t, \mathbf{a}_{t+1}^s, \mathbf{a}_{t+1}^m\}_{t=0}^\infty$ that maximizes (1), taking as given the money supply process $\{M_t\}_{t=0}^\infty$, the bargaining protocol, and the sequence of price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$. For each t , each element of the plan, c_t (fruit consumption), \mathbf{x}_t (production of general goods minus consumption of general goods), $\mathbf{a}_{t+1} = (\mathbf{a}_{t+1}^s, \mathbf{a}_{t+1}^m)$ (equity and money holdings), is a function of the history of dividends and money supplies, and similarly for the price functions.⁷ The plan is *feasible* if it satisfies the initial conditions, and (2) and (3) in every history.⁸ The functional equation (7) is a convenient representation of the agent’s problem. The following result shows that the sequences of solutions for fruit

⁶ Note that S is twice differentiable almost everywhere, with $S'(x) \geq 0$ and $S''(x) \leq 0$ (both inequalities are strict for $x < q^*$), and that $\frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial \lambda_t^s} = \lambda_t^s$ if $\lambda_t \mathbf{a}_t < q^*$, $\frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial \lambda_t^s} = 0$ if $\lambda_t \mathbf{a}_t \geq q^*$, and $\frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial \lambda_t^m} \lambda_t^s = \frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial \lambda_t^s} \lambda_t^m$.

⁷ See Appendix A for a formal description of the time-0 infinite-horizon problem.

⁸ For each history, given c_t , \mathbf{a}_{t+1}^s , and \mathbf{a}_{t+1}^m , the money supply process $\{M_t\}_{t=0}^\infty$, and the price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$, the net production of general goods, \mathbf{x}_t , is immediate from (2), so it will be left implicit in the definition of equilibrium and the analysis hereafter.

consumption and asset holdings induced by the maximization problems in (7) for $t = 0, 1, \dots$ that satisfy certain boundedness conditions, solve the agent’s time-0 optimization problem.

Proposition 1. *Given a money supply process $\{M_t\}_{t=0}^\infty$ and a sequence of price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$, a feasible plan $\{c_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty$ is optimal for the agent from $t = 0$, given initial conditions $\mathbf{a}_0 = (a_0^s, a_0^m)$ and d_0 , if and only if*

$$U'(c_t) - \frac{1}{w_t} \leq 0, \text{ “=” if } c_t > 0, \tag{8}$$

$$-\frac{1}{w_t} \phi_t^s + \beta E_t \{ [1 + S'(\lambda_{t+1} a_{t+1})] \lambda_{t+1}^s \} \leq 0, \text{ “=” if } a_{t+1}^s > 0, \tag{9}$$

$$-\lambda_t^m + \beta E_t \{ [1 + S'(\lambda_{t+1} a_{t+1})] \lambda_{t+1}^m \} \leq 0, \text{ “=” if } a_{t+1}^m > 0, \tag{10}$$

$$\liminf_{t \rightarrow \infty} E_0 \left[\beta^t \frac{1}{w_t} \phi_t^s a_{t+1}^s \right] = 0, \tag{11}$$

$$\liminf_{t \rightarrow \infty} E_0 \left[\beta^t \frac{1}{w_t} \phi_t^m a_{t+1}^m \right] = 0. \tag{12}$$

Definition 1. Given a money supply process $\{M_t\}_{t=0}^\infty$, an equilibrium is a plan $\{c_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty$, pricing functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$, and bilateral terms of trade $\{q_t, p_t\}_{t=0}^\infty$ such that: (i) given prices and the bargaining protocol, $\{c_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty$ solves the agent’s optimization problem; (ii) the terms of trade are determined by Nash bargaining, i.e., $q_t = \min(\lambda_t a_t, q^*)$ and $\lambda_t p_t = q_t$; and (iii) the centralized market clears, i.e., $c_t = d_t$, and $a_{t+1}^s = 1$. The equilibrium is monetary if $\phi_t^m > 0$ for all t , and in this case the money-market clearing condition is $a_{t+1}^m = M_{t+1}$.

The market-clearing conditions immediately give the equilibrium allocations

$$\{c_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty = \{d_t, 1, M_{t+1}\}_{t=0}^\infty,$$

(8) implies $w_t = 1/U'(d_t)$, and once $\{\phi_t^s, \phi_t^m\}_{t=0}^\infty$ has been found, $\{q_t, p_t\}_{t=0}^\infty = \{\lambda_t p_t\}_{t=0}^\infty = \{\min(\Lambda_{t+1}, q^*)\}_{t=0}^\infty$, where $\Lambda_{t+1} \equiv \lambda_{t+1}^s + \lambda_{t+1}^m M_{t+1}$. Therefore, given a money supply process $\{M_t\}_{t=0}^\infty$, and letting $L(\Lambda_{t+1}) \equiv [1 + S'(\Lambda_{t+1})]$, a monetary equilibrium can be summarized by a sequence $\{\phi_t^s, \phi_t^m\}_{t=0}^\infty$ that satisfies

$$U'(d_t) \phi_t^s = \beta E_t [L(\Lambda_{t+1}) U'(d_{t+1}) (\phi_{t+1}^s + d_{t+1})], \tag{13}$$

$$\lambda_t^m = \beta E_t [L(\Lambda_{t+1}) \lambda_{t+1}^m], \tag{14}$$

$$\liminf_{t \rightarrow \infty} E_0 [\beta^t U'(d_t) \phi_t^s] = 0, \tag{15}$$

$$\liminf_{t \rightarrow \infty} E_0 [\beta^t \lambda_t^m M_{t+1}] = 0. \tag{16}$$

There are two assets in this model: equity shares and fiat money. However, to state the results that follow, it will be convenient to be able to refer to a notion of *nominal interest rate*. In order to derive an expression for the “shadow” nominal interest rate, imagine there existed an additional asset in this economy, an *illiquid nominal bond*, i.e., a one-period risk-free bond that pays a unit of money in the centralized market, and which cannot be used in decentralized exchange. Let ϕ_t^n denote the price of this nominal bond. In equilibrium, this price must satisfy $U'(d_t) \phi_t^n =$

$\beta E_t[U'(d_{t+1})\phi_{t+1}^m]$. Since ϕ_t^n/ϕ_t^m is the dollar price of a nominal bond, $i_t = \frac{\phi_t^m}{\phi_t^n} - 1$ is the nominal interest rate. In a monetary equilibrium,

$$i_t = \frac{E_t[L(\Lambda_{t+1})\lambda_{t+1}^m]}{E_t(\lambda_{t+1}^m)} - 1. \tag{17}$$

4. Optimal monetary policy

In this section I consider the problem of choosing an optimal monetary policy. The Pareto optimal allocation can be found by solving the problem of a social planner who wishes to maximize average (equally-weighted across agents) expected utility. The planner chooses a contingent plan $\{c_t, q_t, n_t, y_t, h_t\}_{t=0}^\infty$ subject to the resource constraints, i.e., $y_t \leq h_t$, and $q_t \leq n_t$ for those agents who are matched in the first subperiod of period t , and $q_t = n_t = 0$ for those agents who are not. After imposing these constraints (with equality, with no loss of generality), the planner’s problem becomes

$$\max_{\{c_t, q_t\}_{t=0}^\infty} \liminf_{T \rightarrow \infty} E_0 \left\{ \sum_{t=0}^T \beta^t \{ \alpha [u(q_t) - q_t] + U(c_t) \} \right\}$$

subject to $0 \leq q_t$, and $0 \leq c_t \leq d_t$, and the initial condition $d_0 \in \mathcal{E}$. Here, E_0 denotes the expectation with respect to the probability measure over sequences of dividend realizations induced by $\{F_t\}_{t=0}^\infty$. The solution is to set $\{c_t, q_t\}_{t=0}^\infty = \{d_t, q^*\}_{t=0}^\infty$. From Definition 1, it is clear that the equilibrium consumption of fruit always coincides with the efficient allocation. However, the equilibrium allocation has $q_t \leq q^*$, which may hold with strict inequality in some states. That is, in general, consumption and production in the decentralized market may be too low in a monetary equilibrium.⁹

Proposition 2. *Equilibrium quantities in a monetary equilibrium are Pareto optimal if and only if $i_t = 0$ almost surely (a.s.) for all t .*

The following proposition, establishes two results. The first, is that a deterministic money supply process can suffice to implement a zero nominal rate in every state of the world, even though liquidity needs are stochastic in this environment (because equity, whose value is stochastic, can be used alongside money as means of payment). The second, is that even within the class of deterministic monetary policies, there is a large family of policies that implement the Pareto optimal equilibrium (i.e., there exists a monetary equilibrium with zero nominal rates in every state of the world under the policy). Versions of the second result have been proven in [2,12] for deterministic competitive economies with cash-in-advance constraints that are imposed on agents every period with probability one. Before stating the proposition, it is convenient to introduce some notation. Let $\lambda_t^{s*} = U'(d_t)(\phi_t^{s*} + d_t)$, where

$$\phi_t^{s*} = E_t \sum_{j=1}^\infty \beta^j \frac{U'(d_{t+j})}{U'(d_t)} d_{t+j}, \tag{18}$$

and let \mathcal{T} denote the set of dates, t , for which $q^* - \lambda_t^{s*} > 0$ holds with probability $\pi_t > 0$.

⁹ This is a standard result in the literature, see [9].

Proposition 3. Assume that $\inf_{t \in \mathcal{T}} \pi_t > 0$. A monetary equilibrium with $i_t = 0$ a.s. for all t exists under a deterministic money supply process $\{M_t\}_{t=0}^\infty$ if and only if the following two conditions hold:

$$\liminf_{t \rightarrow \infty} M_t = 0, \tag{19}$$

$$\inf_{t \in \mathcal{T}} M_t \beta^{-t} > 0 \text{ if } \mathcal{T} \neq \emptyset. \tag{20}$$

Conditions (19) and (20) are rather unrestrictive asymptotic conditions. The first one requires that the money supply be arbitrarily close to zero for an infinite number of dates, or equivalently, that there exists some subsequence of dates $\{t_1, t_2, \dots\}$, such that $\lim_{n \rightarrow \infty} M_{t_n} = 0$. The second condition requires that asymptotically, on average over the set of dates \mathcal{T} when fiat money plays an essential role, the growth rate of the money supply must be at least as large as the rate of time preference.¹⁰

The simple class of policies of contracting the money supply at a constant rate, e.g., $M_t = \gamma^t M_0$ for $\gamma \in [\beta, 1)$, satisfies (19) and (20), and hence is consistent with a monetary equilibrium with zero nominal rates in every state. But many other policies are as well. For example, for $b > 0$ sufficiently small, consider

$$M_{t+1} = \gamma^t [1 + b \sin(t)] M_0 \tag{21}$$

for any $\gamma \in (\beta, 1)$. Under (21) the money supply follows deterministic cycles of expansion and contraction forever, and may even contract at a rate larger than β infinitely often, yet this policy implements a monetary equilibrium where the nominal rate is zero in all states.

Consider a deterministic policy $\{M_t\}_{t=0}^\infty$ that satisfies the conditions in Proposition 3. Then, there is a monetary equilibrium with $\phi_t^s = \phi_t^{s*}$ and $\lambda_t^m = \beta^{-t} \lambda_0^m$, where $\lambda_0^m > 0$ is a constant that can be chosen arbitrarily, subject to the additional restriction that

$$\lambda_0^m \geq \frac{q^* - \lambda_t^{s*}}{M_t \beta^{-t}} \text{ for all } t \in \mathcal{T}.$$

In other words, the value of money, λ_t^m , and hence the price level (e.g., the nominal price of fruit, $1/\phi_t^m$, and the nominal price of general goods, $1/[U'(d_t)\phi_t^m]$) is indeterminate under an optimal monetary policy. In this monetary equilibrium, the inflation rate (e.g., in the price of general goods) is

$$\frac{\lambda_t^m}{\lambda_{t+1}^m} = \beta,$$

which is independent of the path of the money supply. This means that the monetary equilibrium with zero nominal interest rates just described, could be obtained, for instance, both under $M_{t+1} = \beta^t M_0$ or under (21), but the inflation rate is the same regardless of which monetary policy is actually followed. Proposition 3 implies that the quantity theory is in general not valid under an optimal monetary policy. (In the equilibrium just described, the quantity theory would no be falsified, however, if the monetary policy was $M_t = \beta^t M_0$.) This feature of an optimal

¹⁰ To get some intuition on (20), consider an economy with $\pi_t = 1$ for all t under an arbitrary deterministic monetary policy, i.e., suppose that $\{\mu_t\}_{t=0}^\infty$ is a positive sequence of real numbers such that $M_t = (\bar{\mu}_t)^t M_0$, where $\bar{\mu}_t = (\prod_{i=0}^{t-1} \mu_i)^{1/t}$ is the geometric average of the growth rate of the money supply through time t . In this case, condition (20) is equivalent to $\liminf_{t \rightarrow \infty} (\bar{\mu}_t / \beta)^t > 0$.

monetary policy was emphasized in [2] in the context of a deterministic cash-in-advance economy.

Assume away the Lucas trees, and this economy reduces to the economy in [9] with buyer-takes-all bargaining. In that paper and in all the subsequent literature, the monetary policy analysis focuses exclusively on equilibria where real money balances are constant, and the money supply grows or declines at a constant rate.¹¹ The usual finding is that the policy of contracting the money supply at a constant rate equal to the rate of time preference is optimal.¹² This policy satisfies conditions (19) and (20), and in an equilibrium with constant real balances, it implies that $M_{t+1}/M_t = \phi_t^m/\phi_{t+1}^m = \beta$, and that real balances equal $\phi_t^m M_t = kq^*$ for all t , so $\phi_t^m = kq^*/M_t$ for all t , where $k \geq 1$ is an arbitrary constant. But as a simple corollary of Proposition 3, this is not the only optimal monetary policy in that model. For example, the policy $M_{t+1}/M_t = \gamma \in (\beta, 1)$ (together with the given M_0), is also an optimal policy, since it is consistent with a monetary equilibrium with zero nominal rates, e.g., $\phi_t^m = \beta^{-t} kq^*/M_0$, and real balances $\phi_t^m M_t = (\gamma/\beta)^t kq^*$ that are growing over time ($k \geq 1$ is again an arbitrary constant). The family of monetary policies that implement equilibria where nominal interest rates are zero is large. However, if one chooses to constrain the set of policies to those where the money supply grows at a constant rate, and to restrict attention to equilibria with constant real money balances, then $M_{t+1}/M_t = \beta$ is the only policy in the family defined by conditions (19) and (20).

5. Conclusion

I have formulated a fairly general version of a prototypical search-based monetary model in which money coexists with equity—a financial asset that yields a risky real return. In this formulation, money is not assumed to be the only asset that must, nor the only asset that can, play the role of a medium of exchange: nothing in the environment prevents agents from using equity along with money, or instead of money, as means of payment. Since the equity share is a claim to a risky aggregate endowment, the fact that agents can use equity to finance purchases implies that they face aggregate liquidity risk, in the sense that in some states of the world, the value of equity holdings may turn out to be too low relative to what would be needed to carry out the transactions that require a medium of exchange. This is a natural context to study the role of money and monetary policy in providing liquidity to lubricate the mechanism of exchange. The model could be augmented to include other types of aggregate uncertainty. For example, one could incorporate aggregate productivity shocks to the technology used to produce general goods, but the main results would not be affected. (The formulation that I have studied is isomorphic to one with aggregate productivity shocks to the technology used to produce special goods.)

An implication of Proposition 3 is that even in a simple deterministic economy, it would be impossible for someone with access to a finite time-series for the path of the money supply, to determine whether an optimal monetary policy is being followed. On the other hand, a single observation of a positive nominal rate would be definitive evidence of a deviation from an optimal monetary policy. According to Proposition 3, there is a large family of monetary policies that are necessary and sufficient to *weakly implement* zero nominal interest rates, in the sense that every policy in the family is consistent with the existence of a monetary equilibrium with

¹¹ Lagos and Wright [8] analyze dynamic equilibria with real balances that vary over time, but do not study monetary policy (the money supply is kept constant to focus on dynamics due exclusively to beliefs).

¹² This policy implements $q_t = q^*$ in every meeting with buyer-takes-all bargaining. Otherwise, the policy is still optimal among the feasible class of policies considered, but implements $q_t < q^*$.

zero nominal interest rates. This result leaves open for future research the question of *unique implementation of monetary equilibrium*: Are there monetary policies in the optimal class under which the equilibrium with zero nominal rates at all dates and in all states is the unique monetary equilibrium? It is not difficult to find policies that implement zero nominal rates weakly but not uniquely.¹³ Notice that even if one were able to find a family of policies that imply that a monetary equilibrium with zero nominal rates exists, and that it is the unique monetary equilibrium, the question of *unique implementation of equilibrium* would still require one to deal with the fact that a nonmonetary equilibrium will always exist.

Throughout the paper I have emphasized the similarities between my results and those that have been established for competitive economies subject to cash-in-advance constraints, so at this point I should perhaps comment on some of the differences. The analogous results for cash-in-advance economies, e.g., those in [2] and [12], have been established in environments with no aggregate uncertainty, and where it is assumed that money must be used in order to purchase a certain good (with probability one in every period). In contrast, I have proved Propositions 2 and 3 for search-based environments with aggregate uncertainty, and where money may not be used or needed as a medium of exchange in some periods, for some realizations of the aggregate uncertainty.

An implication of Proposition 3 is that even if agent's liquidity needs are stochastic, one need not look beyond the class of deterministic monetary policies to implement the optimum. In particular, it is not necessary to resort to (state-contingent) monetary policy rules if the purpose is to implement zero nominal rates. To some, the failure of the quantity theory and the ensuing price-level indeterminacy may seem unappealing if the model is to be used for applied research. One way to address these concerns, would be to characterize the family of policies that implement a constant but positive nominal interest rate, and then study the behavior of the limiting economy as the target nominal rate approaches zero. The optimal prescription for monetary policy—the Friedman rule—requires: (a) that the nominal interest be constant, and (b) that this constant be zero, so this class of non-optimal policies would represent a perturbation of the Friedman rule along the second dimension. A policy that targets a constant nonzero nominal rate in this stochastic environment, however, will typically have to implement stochastic real balances, which could require a stochastic monetary policy rule. I study these issues in [6].

Appendix A

I begin with a formal description of the time-zero infinite-horizon problem faced by an agent in the monetary economy. Let $\omega = \{d_t, M_t\}_{t=0}^{\infty}$ denote a realization of dividends and money supplies, and let Ω be the set of all such realizations. Let $\omega^t = \{d_k, M_k\}_{k=0}^t$ denote a history of dividends and money supplies up to time t , and let Ω^t be the collection of all such histories. Consider the probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where \mathcal{H} is an appropriate σ -field of subsets of Ω (e.g., the σ -field generated by Ω^t for all finite t), and \mathbb{P} is the probability measure on \mathcal{H} induced by the transition functions $\{F_t\}_{t=0}^{\infty}$ and the monetary policy $\{\mu_t\}_{t=0}^{\infty}$. Let $\mathcal{H}^t \subseteq \mathcal{H}$ be a partition of Ω such that $H_{\omega}^t \in \mathcal{H}^t$ is a set of histories that coincide until time t , i.e., $H_{\omega}^t = \{\omega \in \Omega : \omega^t =$

¹³ For example, in the pure monetary economy discussed in the end of Section 4, I pointed out that the monetary policy $M_{t+1}/M_t = \gamma \in (\beta, 1)$ for all t , is consistent with a monetary equilibrium with zero nominal rates and real balances that grow over time. However, under this monetary policy there also exists a monetary equilibrium with constant real balances $\bar{z} \in (0, q^*)$ and a constant nominal interest rate $i = \alpha[u'(\bar{z}) - 1] > 0$. (This last equilibrium exists provided $u'(0) > 1 + \frac{\gamma - \beta}{\alpha\beta}$, and in this case, \bar{z} solves $u'(\bar{z}) = 1 + \frac{\gamma - \beta}{\alpha\beta}$.)

$\bar{\omega}$ for some $\bar{\omega} \in \Omega^t$. The σ -field generated by \mathcal{H}^t , denoted \mathcal{F}_t , captures the information available to the investor at time t , and the filtration $\{\mathcal{F}_t\}_{t=0}^\infty$ represents how this information is revealed over time.

At $t = 0$, the agent takes as given the sequence of \mathcal{F}_t -measurable price functions,

$$\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty,$$

and the sequence of \mathcal{F}_t -measurable monetary policy functions, $\{\mu_t\}_{t=0}^\infty$, where $w_t : \Omega \rightarrow \mathbb{R}^+$, $\phi_t^s : \Omega \rightarrow \mathbb{R}^+$, $\phi_t^m : \Omega \rightarrow \mathbb{R}^+$, and $\mu_{t+1} : \Omega \rightarrow \mathbb{R}^+$. From (5), notice that $\lambda_t^s : \Omega \rightarrow \mathbb{R}^+$ and $\lambda_t^m : \Omega \rightarrow \mathbb{R}^+$ are \mathcal{F}_t -measurable functions as well. A *feasible plan*, $\chi = \{c_t, x_t, a_t^s, a_t^m\}_{t=0}^\infty$, is a value $(a_0^s, a_0^m) = (a_0^s, a_0^m) \in \mathbb{R}^+ \times \mathbb{R}^+$ and a sequence of \mathcal{F}_t -measurable functions

$$\{c_t, x_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty,$$

where $c_t : \Omega \rightarrow \mathbb{R}^+$, $a_{t+1}^s : \Omega \rightarrow \mathbb{R}^+$, $a_{t+1}^m : \Omega \rightarrow \mathbb{R}^+$, and

$$x_t = \frac{1}{w_t} [(\phi_t^s + d_t)a_t^s + \phi_t^m(a_t^m + T_t) - \phi_t^s a_{t+1}^s - \phi_t^m a_{t+1}^m - c_t],$$

with $T_t = M_{t+1} - M_t$. Let \mathbb{A} denote the set of all feasible plans.

Let $\mathcal{U}^T(\cdot, \mathbf{a}_0, \omega_0)$ be the utility functional for the agent from $t = 0$ until $t = T$, given that $\mathbf{a}_0 = (a_0^s, a_0^m)$ is the agent's initial portfolio, and $\omega_0 = (d_0, M_0)$ is the initial condition for the dividend and the money supply. The agent's utility from following a feasible policy χ over this period, taking as given the sequence of price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$, is

$$\begin{aligned} \mathcal{U}^T(\chi, \mathbf{a}_0, \omega_0) = & E_0 \left\{ \sum_{t=0}^T \beta^t \left[\mathcal{S}(\lambda_t \mathbf{a}_t) + \lambda_t \mathbf{a}_t - \frac{1}{w_t} \phi_t \mathbf{a}_{t+1} \right] \right\} \\ & + E_0 \left\{ \sum_{t=0}^T \beta^t \left[U(c_t) - \frac{1}{w_t} c_t \right] \right\} + K_T, \end{aligned}$$

where $K_T \equiv E_0 \{ \sum_{t=0}^T \beta^t \frac{\phi_t^m}{w_t} T_t \}$. The notation E_t is shorthand for the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_t]$. With the Law of Iterated Expectations, $\mathcal{U}^T(\chi, \mathbf{a}_0, \omega_0)$ can be rearranged as follows:

$$\begin{aligned} \mathcal{U}^T(\chi, \mathbf{a}_0, \omega_0) = & \mathcal{S}(\lambda_0 \mathbf{a}_0) + \lambda_0 \mathbf{a}_0 + K_T + E_0 \left\{ \sum_{t=0}^T \beta^t \left[U(c_t) - \frac{1}{w_t} c_t \right] \right\} \\ & + E_0 \left\{ \sum_{t=0}^{T-1} \beta^t \left[\beta E_t \mathcal{S}(\lambda_{t+1} \mathbf{a}_{t+1}) - \left(\frac{1}{w_t} \phi_t^s - \beta E_t \lambda_{t+1}^s \right) a_{t+1}^s \right. \right. \\ & \left. \left. - (\lambda_t^m - \beta E_t \lambda_{t+1}^m) a_{t+1}^m \right] \right\} \\ & - E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s a_{T+1}^s + \phi_T^m a_{T+1}^m] \right\}. \end{aligned} \tag{22}$$

The utility $\mathcal{U}^T(\cdot, \mathbf{a}_0, \omega_0)$ associated with the agent's problem has been defined for an arbitrary sequence of \mathcal{F}_t -measurable price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$. Some price functions will be inconsistent with an equilibrium, so there is no loss in restricting the analysis of the agent's problem to a family of functions that excludes such functions. In particular, there is no loss in restricting

the analysis to *admissible* price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$, namely price functions that satisfy the no-arbitrage conditions, $\beta E_t \lambda_{t+1}^s - \frac{1}{w_t} \phi_t^s \leq 0$, and $\beta E_t \lambda_{t+1}^m - \lambda_t^m \leq 0$ for all t .

Next, define the infinite-horizon utility for the agent from following a feasible plan χ , by $\mathcal{U}(\chi, \mathbf{a}_0, \omega_0) = \liminf_{T \rightarrow \infty} \mathcal{U}^T(\chi, \mathbf{a}_0, \omega_0)$.

Lemma 1. Given admissible price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$,

$$\begin{aligned} \mathcal{U}(\chi, \mathbf{a}_0, \mathbf{s}_0) &= \mathcal{S}(\lambda_0 \mathbf{a}_0) + \lambda_0 \mathbf{a}_0 + \liminf_{T \rightarrow \infty} K_T + E_0 \left\{ \sum_{t=0}^\infty \beta^t \left[U(\mathbf{c}_t) - \frac{1}{w_t} \mathbf{c}_t \right] \right\} \\ &+ E_0 \left\{ \sum_{t=0}^\infty \beta^t \left[\beta E_t \mathcal{S}(\lambda_{t+1} \mathbf{a}_{t+1}) - \left(\frac{1}{w_t} \phi_t^s - \beta E_t \lambda_{t+1}^s \right) \mathbf{a}_{t+1}^s \right. \right. \\ &\left. \left. - (\lambda_t^m - \beta E_t \lambda_{t+1}^m) \mathbf{a}_{t+1}^m \right] \right\} \\ &- \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\}. \end{aligned} \tag{23}$$

Proof. Let $S_T = \sum_{t=0}^T \beta^t U(\mathbf{c}_t) + \sum_{t=0}^{T-1} \beta^{t+1} E_t \mathcal{S}(\lambda_{t+1} \mathbf{a}_{t+1})$. U and \mathcal{S} are nondecreasing, with $U(0) = \mathcal{S}(0) = 0$, so $U(\mathbf{c}_t) \geq 0$, and $\mathcal{S}(\lambda_{t+1} \mathbf{a}_{t+1}) \geq 0$ for all t , and therefore $\{S_T\}_{T=0}^\infty$ is a nondecreasing sequence of nonnegative (extended) real-valued measurable functions, and it has a limit, i.e., $\lim_{T \rightarrow \infty} S_T = \sum_{t=0}^\infty \beta^t [U(\mathbf{c}_t) + \beta E_t \mathcal{S}(\lambda_{t+1} \mathbf{a}_{t+1})]$. Then by the Monotone Convergence Theorem [11, Theorem 7.8],

$$\liminf_{T \rightarrow \infty} E_0 S_T = \lim_{T \rightarrow \infty} E_0 S_T = E_0 \lim_{T \rightarrow \infty} S_T. \tag{24}$$

Let $S'_T = \sum_{t=0}^T \beta^t \frac{1}{w_t} \mathbf{c}_t + \sum_{t=0}^{T-1} \beta^{t+1} [(\frac{1}{w_t} \phi_t^s - \beta E_t \lambda_{t+1}^s) \mathbf{a}_{t+1}^s + (\lambda_t^m - \beta E_t \lambda_{t+1}^m) \mathbf{a}_{t+1}^m]$. Since each term in this partial sum is nonnegative, $\{S'_T\}$ is a nondecreasing sequence, so $\lim_{T \rightarrow \infty} S'_T = \sum_{t=0}^\infty \beta^t [\frac{1}{w_t} \mathbf{c}_t + (\frac{1}{w_t} \phi_t^s - \beta E_t \lambda_{t+1}^s) \mathbf{a}_{t+1}^s + (\lambda_t^m - \beta E_t \lambda_{t+1}^m) \mathbf{a}_{t+1}^m]$ exists, although it may be $+\infty$. Then by the Monotone Convergence Theorem,

$$\liminf_{T \rightarrow \infty} E_0 S'_T = \lim_{T \rightarrow \infty} E_0 S'_T = E_0 \lim_{T \rightarrow \infty} S'_T. \tag{25}$$

With (24) and (25), take $\liminf_{T \rightarrow \infty}$ on both sides of (22) to arrive at (23). \square

At $t = 0$, the agent takes as given the initial conditions \mathbf{a}_0 and ω_0 , and a sequence of price functions $\{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$, and solves

$$\max_{\chi \in \mathbb{A}} \mathcal{U}(\chi, \mathbf{a}_0, \omega_0).$$

Proposition 1 characterizes the optimal plan (termed the *maximal* plan in [1]).

Proof of Proposition 1. Step (i): First show that (8)–(12) are sufficient for an optimum. Let $\chi = \{\mathbf{c}_t, \mathbf{x}_t, \mathbf{a}_{t+1}^s, \mathbf{a}_{t+1}^m\}_{t=0}^\infty$ be the plan that satisfies (8)–(12), and $\tilde{\chi} = \{\tilde{\mathbf{c}}_t, \tilde{\mathbf{x}}_t, \tilde{\mathbf{a}}_{t+1}^s, \tilde{\mathbf{a}}_{t+1}^m\}_{t=0}^\infty$ be any other feasible plan. Let $\Delta \equiv \mathcal{U}(\chi, \mathbf{a}_0, \omega_0) - \mathcal{U}(\tilde{\chi}, \mathbf{a}_0, \omega_0)$, then

$$\begin{aligned} \Delta = & E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ U(\mathbf{c}_t) - U(\tilde{\mathbf{c}}_t) - \frac{1}{w_t} (\mathbf{c}_t - \tilde{\mathbf{c}}_t) \right\} \right\} \\ & + E_0 \left\{ \sum_{t=0}^{\infty} \beta^{t+1} E_t [S(\lambda_{t+1} \mathbf{a}_{t+1}) - S(\lambda_{t+1} \tilde{\mathbf{a}}_{t+1})] \right\} \\ & - E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{w_t} \phi_t^s - \beta E_t \lambda_{t+1}^s \right) (\mathbf{a}_{t+1}^s - \tilde{\mathbf{a}}_{t+1}^s) \right\} \\ & - E_0 \left\{ \sum_{t=0}^{\infty} \beta^t (\lambda_t^m - \beta E_t \lambda_{t+1}^m) (\mathbf{a}_{t+1}^m - \tilde{\mathbf{a}}_{t+1}^m) \right\} \\ & + \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \tilde{\mathbf{a}}_{T+1}^s + \phi_T^m \tilde{\mathbf{a}}_{T+1}^m] \right\} \\ & - \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\}. \end{aligned}$$

Since U and S are concave and differentiable, $U(\mathbf{c}_t) - U(\tilde{\mathbf{c}}_t) \geq U'(\mathbf{c}_t)(\mathbf{c}_t - \tilde{\mathbf{c}}_t)$, and

$$\begin{aligned} & S(\lambda_{t+1} \mathbf{a}_{t+1}) + \frac{\partial S(\lambda_{t+1} \mathbf{a}_{t+1})}{\partial \mathbf{a}_{t+1}^s} (\tilde{\mathbf{a}}_{t+1}^s - \mathbf{a}_{t+1}^s) \\ & + \frac{\partial S(\lambda_{t+1} \mathbf{a}_{t+1})}{\partial \mathbf{a}_{t+1}^m} (\tilde{\mathbf{a}}_{t+1}^m - \mathbf{a}_{t+1}^m) \geq S(\lambda_{t+1} \tilde{\mathbf{a}}_{t+1}), \end{aligned}$$

with $\frac{\partial S(\lambda_{t+1} \mathbf{a}_{t+1})}{\partial \mathbf{a}_{t+1}^i} = S'(\lambda_{t+1} \mathbf{a}_{t+1}) \lambda_{t+1}^i$ for $i = s, m$, so

$$\begin{aligned} \Delta \geq & E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[U'(\mathbf{c}_t) - \frac{1}{w_t} \right] (\mathbf{c}_t - \tilde{\mathbf{c}}_t) \right\} \\ & + E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t [1 + S'(\lambda_{t+1} \mathbf{a}_{t+1})] \lambda_{t+1}^s - \frac{1}{w_t} \phi_t^s \right\} (\mathbf{a}_{t+1}^s - \tilde{\mathbf{a}}_{t+1}^s) \right\} \\ & + E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t [1 + S'(\lambda_{t+1} \mathbf{a}_{t+1})] \lambda_{t+1}^m - \lambda_t^m \right\} (\mathbf{a}_{t+1}^m - \tilde{\mathbf{a}}_{t+1}^m) \right\} \\ & + \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \tilde{\mathbf{a}}_{T+1}^s + \phi_T^m \tilde{\mathbf{a}}_{T+1}^m] \right\} \\ & - \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\}. \end{aligned}$$

With (8)–(10) and the fact that $\tilde{\mathbf{c}}_t \geq 0$, $\tilde{\mathbf{a}}_{t+1}^s \geq 0$, and $\tilde{\mathbf{a}}_{t+1}^m \geq 0$ (because $\tilde{\chi}$ is feasible), the previous inequality implies

$$\begin{aligned} \Delta \geq & E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[U'(\mathbf{c}_t) - \frac{1}{w_t} \right] \mathbf{c}_t \right\} \\ & + E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t [1 + S'(\lambda_{t+1} \mathbf{a}_{t+1})] \lambda_{t+1}^s - \frac{1}{w_t} \phi_t^s \right\} \mathbf{a}_{t+1}^s \right\} \end{aligned}$$

$$\begin{aligned}
 &+ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \{ \beta E_t [1 + S'(\lambda_{t+1} \mathbf{a}_{t+1})] \lambda_{t+1}^m - \lambda_t^m \} \mathbf{a}_{t+1}^m \right\} \\
 &+ \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \tilde{\mathbf{a}}_{T+1}^s + \phi_T^m \tilde{\mathbf{a}}_{T+1}^m] \right\} \\
 &- \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\}.
 \end{aligned}$$

Use (8)–(10) once again, to obtain

$$\begin{aligned}
 \Delta &\geq \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \tilde{\mathbf{a}}_{T+1}^s + \phi_T^m \tilde{\mathbf{a}}_{T+1}^m] \right\} \\
 &- \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\} \\
 &\geq - \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\}.
 \end{aligned}$$

With (11) and (12), this last inequality implies $\Delta \geq 0$, so the plan χ is optimal.

Step (ii): Next, show that an optimal plan, $\chi = \{\mathbf{c}_t, \mathbf{x}_t, \mathbf{a}_{t+1}^s, \mathbf{a}_{t+1}^m\}_{t=0}^{\infty}$, must satisfy (8)–(12).

Since χ is an optimal plan, (23) implies that for each t , $\mathbf{c}_t = \arg \max_{c \geq 0} [U(c) - \frac{1}{w_t} c]$, and

$$\begin{aligned}
 \mathbf{a}_{t+1} \in \arg \max_{\tilde{\mathbf{a}}_{t+1} \geq 0} \left\{ -\frac{1}{w_t} \phi_t^s \tilde{\mathbf{a}}_{t+1}^s - \lambda_t^m \tilde{\mathbf{a}}_{t+1}^m + \beta E_t [S(\lambda_{t+1} \tilde{\mathbf{a}}_{t+1}) \right. \\
 \left. + \lambda_{t+1}^s \tilde{\mathbf{a}}_{t+1}^s + \lambda_{t+1}^m \tilde{\mathbf{a}}_{t+1}^m] \right\}.
 \end{aligned}$$

Since both U and S are differentiable, $\{\mathbf{c}_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$ must satisfy (8)–(10). To show that (11) and (12) are necessary for an optimum, use the optimal plan $\chi = \{\mathbf{c}_t, \mathbf{x}_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$ to construct the feasible plan $\chi^\varepsilon = \{\mathbf{c}_t, \mathbf{x}_t^\varepsilon, (1 - \varepsilon)\mathbf{a}_{t+1}\}_{t=0}^{\infty}$, for some small $\varepsilon > 0$, where

$$\mathbf{x}_0^\varepsilon = \frac{1}{w_0} [(\phi_0^s + d_0)\mathbf{a}_0^s + \phi_0^m (\mathbf{a}_0^m + T_0) - (1 - \varepsilon)(\phi_0^s \mathbf{a}_1^s + \phi_0^m \mathbf{a}_1^m) - \mathbf{c}_0],$$

and

$$\mathbf{x}_t^\varepsilon = \frac{1}{w_t} \{ (1 - \varepsilon)[(\phi_t^s + d_t)\mathbf{a}_t^s + \phi_t^m \mathbf{a}_t^m - \phi_t^s \mathbf{a}_{t+1}^s - \phi_t^m \mathbf{a}_{t+1}^m] - \mathbf{c}_t + \phi_t^m T_t \}$$

for $t \geq 1$. Let $\Delta_\varepsilon \equiv \mathcal{U}(\chi, \mathbf{a}_0, \omega_0) - \mathcal{U}(\chi^\varepsilon, \mathbf{a}_0, \omega_0)$; then,

$$\begin{aligned}
 \Delta_\varepsilon &= E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t [S(\lambda_{t+1} \mathbf{a}_{t+1}) - S((1 - \varepsilon)\lambda_{t+1} \mathbf{a}_{t+1})] \right. \right. \\
 &- \varepsilon \left[\frac{1}{w_t} \phi_t^s \mathbf{a}_{t+1}^s + \lambda_t^m \mathbf{a}_{t+1}^m - \beta E_t \lambda_{t+1} \mathbf{a}_{t+1} \right] \left. \right\} \\
 &- \varepsilon \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s \mathbf{a}_{T+1}^s + \phi_T^m \mathbf{a}_{T+1}^m] \right\}.
 \end{aligned}$$

Divide the previous expression by ε , and take the limit as $\varepsilon \rightarrow 0$ to arrive at

$$\lim_{\varepsilon \rightarrow 0} \frac{\Delta_\varepsilon}{\varepsilon} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{w_t} \phi_t^s a_{t+1}^s - \lambda_t^m a_{t+1}^m + \beta E_t \left\{ [1 + \mathcal{S}'(\lambda_{t+1} a_{t+1})] \lambda_{t+1} a_{t+1} \right\} \right\} \right\} \\ - \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s a_{T+1}^s + \phi_T^m a_{T+1}^m] \right\}.$$

Since the $\{a_{t+1}\}_{t=0}^\infty$ is part of an optimal plan, the first-order conditions (9) and (10) imply

$$\lim_{\varepsilon \rightarrow 0} \frac{\Delta_\varepsilon}{\varepsilon} = - \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s a_{T+1}^s + \phi_T^m a_{T+1}^m] \right\},$$

and the optimality of χ requires

$$0 \leq - \liminf_{T \rightarrow \infty} E_0 \left\{ \beta^T \frac{1}{w_T} [\phi_T^s a_{T+1}^s + \phi_T^m a_{T+1}^m] \right\}.$$

Since $\beta^t \frac{1}{w_t} \phi_t^s a_{t+1}^s \geq 0$ and $\beta^t \frac{1}{w_t} \phi_t^m a_{t+1}^m \geq 0$ for all t , it follows that (11) and (12) must hold. \square

Proof of Proposition 2. Note that $i_t : \Omega \rightarrow \mathbb{R}^+$ is an \mathcal{F}_t -measurable function, and from (17), $i_t(\omega) = 0$ a.s. for all t implies $L(\Lambda_{t+1}) = 1$ a.s. for all t . Then, since

$$L(\Lambda_{t+1}) = \begin{cases} 1 & \text{if } q^* \leq \Lambda_{t+1}, \\ 1 - \alpha + \alpha u'(\Lambda_{t+1}) > 1 & \text{if } \Lambda_{t+1} < q^*, \end{cases} \tag{26}$$

$L(\Lambda_{t+1}) = 1$ a.s. for all t , implies $q_{t+1} = \min(\Lambda_{t+1}, q^*) = q^*$ a.s. for all t . To conclude, note that $q_{t+1} = q^*$ a.s. for all t implies $q^* \leq \Lambda_{t+1}$ a.s. for all t , so (26) implies $L(\Lambda_{t+1}) = 1$ a.s. for all t , and (17) implies $i_t(\omega) = 0$ a.s. for all t . \square

Proof of Proposition 3. Let $\Omega_t^* = \{\omega \in \Omega : q^* - \lambda_t^{s*}(\omega) > 0\}$, and $\mathcal{T} = \{t \in \{0, 1, \dots\} : E_0[\mathbb{I}_{\Omega_t^*}(\omega)] > 0\}$, where $\mathbb{I}_{\Omega_t^*}(\omega)$ is an indicator function that equals 1 if $\omega \in \Omega_t^*$. Note that $E_0[\mathbb{I}_{\Omega_t^*}(\omega)] = \mathbb{P}(\Omega_t^*) \equiv \pi_t$ in the statement of the proposition.

Step 1 (\Leftarrow): Show that if (19) and (20) hold, then there exists a monetary equilibrium with $i_t = 0$ a.s. for all t . Construct the equilibrium as follows. Set $q_t(\omega) = q^*$ for every ω and all t . Then the Euler equations (13) and (14) become

$$U'(d_t) \phi_t^s = \beta E_t[U'(d_{t+1})(\phi_{t+1}^s + d_{t+1})], \tag{27}$$

$$\lambda_t^m = \beta E_t \lambda_{t+1}^m. \tag{28}$$

Let $\phi_t^s = \phi_t^{s*}$, for all ω and t , and notice that it satisfies (15) and (27). Let λ_0^m be a positive constant (it will be determined below), let $\lambda_t^m(\omega) = \beta^{-t} \lambda_0^m$ for all ω and t , and notice that $\{\lambda_{t+1}^m\}_{t=0}^\infty$ satisfies (28). Also,

$$\liminf_{t \rightarrow \infty} E_0[\beta^t \lambda_t^m M_{t+1}] = \liminf_{t \rightarrow \infty} E_0[\lambda_0^m M_{t+1}] = \lambda_0^m \liminf_{t \rightarrow \infty} M_{t+1} = 0,$$

where the last equality follows from condition (19), so (16) is also satisfied. All that remains is to show that λ_0^m can be chosen such that $\lambda_t^m(\omega) > 0$ for all ω and t (so the equilibrium constructed is indeed monetary), and such that $\lambda_t^m(\omega) M_t \geq q^* - \lambda_t^{s*}(\omega)$ a.s. for all t (so that real balances are consistent with $q_t = q^*$, and hence with $i_t = 0$ a.s. for all t). Given that $\lambda_t^m(\omega) = \beta^{-t} \lambda_0^m$, any positive choice of λ_0^m guarantees $\lambda_t^m(\omega) > 0$ for every ω and all t . In particular, if the primitives of the economy are such that $\mathcal{T} = \emptyset$, choose $\lambda_0^m = k \in (0, \infty)$, which implies $\lambda_t^m(\omega) M_t = k M_t \beta^{-t} > 0 \geq q^* - \lambda_t^{s*}(\omega)$ for all ω and t . Conversely, if $\mathcal{T} \neq \emptyset$, choose

$$\lambda_0^m = \frac{q^*}{\inf_{t \in \mathcal{T}} M_t \beta^{-t}} \in (0, \infty). \tag{29}$$

Then

$$\lambda_t^m(\omega)M_t = \frac{M_t\beta^{-t}}{\inf_{t \in \mathcal{T}} M_t\beta^{-t}}q^* \geq q^* - \lambda_t^{s*}(\omega) \quad \text{for all } \omega, t.$$

Step 2 (\Rightarrow): Show that if $\{q_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$ is a monetary equilibrium with $i_t = 0$ a.s. for all t , then (19) and (20) must hold. In such an equilibrium, $\{\lambda_t^m\}$ satisfies (28), which together with the Law of Iterated Expectations, implies

$$\lambda_0^m = E_0\beta^t\lambda_t^m. \tag{30}$$

Also, in any monetary equilibrium,

$$\liminf_{t \rightarrow \infty} E_0[\beta^t\lambda_t^m M_{t+1}] = \lambda_0^m \liminf_{t \rightarrow \infty} M_{t+1} = 0 \tag{31}$$

by (16). Since $\lambda_0^m > 0$ in a monetary equilibrium, (31) implies (19). A monetary equilibrium with $i_t = 0$ a.s. for all t also satisfies

$$\lambda_t^m(\omega)M_t \begin{cases} \geq q^* - \lambda_t^{s*}(\omega) & \text{for all } t \text{ and a.e. } \omega \in \Omega_t^*, \\ > 0 & \text{for all } t \text{ and a.e. } \omega \notin \Omega_t^*. \end{cases} \tag{32}$$

Multiply the top inequality in (32) through by $\mathbb{I}_{\Omega_t^*}(\omega)$, and the bottom inequality by $[1 - \mathbb{I}_{\Omega_t^*}(\omega)]$, and add them to obtain

$$\lambda_t^m(\omega)M_t \geq \mathbb{I}_{\Omega_t^*}(\omega)[q^* - \lambda_t^{s*}(\omega)] \quad \text{for all } t \text{ and a.e. } \omega. \tag{33}$$

For all $t \notin \mathcal{T}$, (33) implies $\lambda_t^m(\omega) \geq 0$ for a.e. ω , merely an equilibrium condition. But (33) also implies

$$\lambda_t^m(\omega)M_t \geq \mathbb{I}_{\Omega_t^*}(\omega)(q^* - \bar{\lambda}^{s*}) \quad \text{for all } t \in \mathcal{T} \text{ and a.e. } \omega,$$

where $\bar{\lambda}^{s*} \equiv \sup_{t \in \mathcal{T}} \sup_{\omega \in \Omega_t^*} \lambda_t^{s*}(\omega)$. Together with (30), this last inequality implies

$$\lambda_0^m M_t \beta^{-t} \geq \pi_t (q^* - \bar{\lambda}^{s*}) \quad \text{for all } t \in \mathcal{T}.$$

This last condition is vacuous if the primitives of the economy are such that $\mathcal{T} = \emptyset$, but implies

$$\inf_{t \in \mathcal{T}} M_t \beta^{-t} \geq \frac{q^* - \bar{\lambda}^{s*}}{\lambda_0^m} \inf_{t \in \mathcal{T}} \pi_t > 0 \quad \text{if } \mathcal{T} \neq \emptyset.$$

Hence, (20) is also necessary in a monetary equilibrium with $i_t = 0$ a.s. for all t . \square

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